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TECHNICAL NOTE

A large-strain radial consolidation theory for soft clays improved by vertical drains

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A system of vertical drains with combined vacuum and surcharge preloading is an effective solution for promoting radial flow, accelerating consolidation. However, when a mixture of soil and water is deposited at a low initial density, a significant amount of deformation or surface settlement occurs. Therefore, it is necessary to introduce large-strain theory, which has been widely used to manage dredged disposal sites in one-dimensional theory, into radial consolidation theory. A governing equation based on Gibson's large-strain theory and Barron's free-strain theory incorporating the radial and vertical flows, the weight of the soil, variable hydraulic conductivity and compressibility during the consolidation process is therefore presented.

KEYWORDS: consolidation; constitutive relations; ground improvement; time dependence

INTRODUCTION

Soft clays often have low bearing capacity, high compressibility with high water content and large void ratios, affecting the long-term stability of buildings, roads, rail tracks and other forms of major infrastructure (Geng *et al.*, 2011). Therefore, it is imperative to stabilise these soils before commencing construction, thereby preventing unacceptable differential settlement. Vertical drains combined with vacuum pressure and surcharge preloading are widely used to accelerate the consolidation of soft clay, decreasing excess pore-water pressure and increasing effective stress (Hansbo, 1979; Atkinson & Eldred, 1981; Runesson *et al.*, 1985; Holtz *et al.*, 1991; Hird *et al.*, 1992; Mesri *et al.*, 1994; Indraratna & Redana, 2000; Zhu & Yin, 2000; Fox *et al.*, 2003; Walker & Indraratna, 2006; Indraratna *et al.*, 2009; Ghandeharioon *et al.*, 2010). Negative suction by vacuum preloading along the length of the drain causes a radial hydraulic flow towards the drain. This, in turn, prevents the build-up of high excess pore-water pressure in the soil by reducing the seepage path. Additionally, less pore-water pressure build-up reduces the risk of failure (Indraratna *et al.*, 2005; Geng *et al.*, 2011, 2012). The theory of Barron (1948) has formed the basis for most analyses and research on radial consolidation, where two types of vertical drains are considered: 'free vertical strain' resulting from a uniform distribution of surface load, and 'equal vertical strain' caused by imposing the same vertical deformation on the surface for a uniform soil. To obtain closed-form solutions, the vertical strain is assumed to be infinitesimal, the weight of the soil is usually omitted, the hydraulic conductivity and coefficient of compressibility of the soil are assumed to remain unchanged from a given load increment and linear stress–strain behaviour is assumed. However, with thick marine soft clay layers, these simplifying assumptions are not applicable. When a mixture of soil and water is deposited at a low initial density, a significant amount of deformation or surface settlement occurs, especially when prefabricated vertical drains (PVDs) and vacuum preloading techniques are used. In cases where an embankment has recently been constructed or the clay is deep and soft because consolidation is incomplete, the weight of the soil cannot be omitted. Furthermore, it has been recognised that the assumption of constant permeability and compressibility does not prove to be acceptable for very soft soil portions which demonstrate highly plastic deformation (Mikasa, 1965; Geng *et al.*, 2006).

Thus, although large strains are commonly encountered in projects where PVD with surcharge or vacuum preloading are used (Kwan Lo & Mesri, 1994; Selfridge & McIntosh, 1994; Bergado *et al.*, 1997), it is rare to find any research conducted on PVDs combined with vacuum preloading and surcharge preloading using large-strain theory (Ito & Azam., 2013). For one-dimensional large-strain theory, Gibson *et al.* (1981) used a continuum model to describe the large-strain consolidation of a layer of soil under its own weight. Since Gibson's publication, experimental and numerical issues of one-dimensional large-strain consolidation have been reported (Znidarcic *et al.*, 1984; Tan *et al.*, 1990; De Boer *et al.*, 1996; Sills, 1998; Toorman, 1999; Bartholomeeusen *et al.*, 2002).

Therefore, on the basis of Gibson *et al.*'s (1981) one-dimensional large-strain theory and Barron's (1948) free strain theory, a governing equation that considers the radial and vertical flows, the weight of the soil, time-dependent surcharge and vacuum preloading, along with variable hydraulic conductivity and compressibility during consolidation, is presented.

MODEL DESCRIPTION AND COORDINATE SYSTEMS

To obtain the governing equation for the large-strain consolidation of soil with vertical drains, the following assumptions are made.

(*a*) A saturated homogeneous layer of soil with an initial height H is treated as an idealised two-phase material,

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in which the solid particles and pore fluid are incompressible. The term 'homogeneous' refers to the constitutive relationships of the layer, not the initial distribution of the void ratio within the layer.

(*b*) The soil is fully saturated.
(*c*) Darcy's law is valid.
(*d*) Load is distributed uniformly over this area.
(*e*) The external radius r_e is impervious, or no flow occurs across this boundary because of symmetry.
(*f*) All compressive strains within the soil mass occur in a vertical direction within the radial velocity of the solid skeleton $v_r^s = 0$.
(*g*) The principal directions of the stress tensor coincide with the vertical (ξ) and radial (r) axes.

As in Barron (1948), the problem is simplified to be axisymmetric, as shown in Fig. 1(a).

In the vertical direction, the usual coordinate system used in geotechnical engineering is the Eulerian system where the deformation of material is related to planes fixed in space. Thus, excess pore pressure in a consolidating layer of clay is measured at a point specifically related to a fixed physical datum. It should be noted that under this system the particles (porous skeleton) move with respect to the Eulerian coordinate system as consolidation proceeds. Under infinitesimal strain, the theories of consolidation assume that the thickness of the compressible layer is constant and any deformation of the layer during consolidation is assumed to be small, compared with its thickness. Using a Eulerian system, a piezometer is located at some point in the layer of clay and is referenced in space to a fixed datum. Figs 1(b) and 1(c) show the Lagrangian and convective coordinates. A saturated layer of clay has an initial thickness H with a fixed bottom, for example, a rock boundary underlying soft clay (Gibson *et al.*, 1981) (Fig. 1(b)). A thin sample of the clay layer ($A_0B_0C_0D_0$) has a coordinate position a and has thickness δa (Gibson *et al.*, 1981). Note that the position of the bottom boundary (datum plane) is at $a = 0$. The position of the upper boundary is at $a = a_0$ (Gibson *et al.*, 1981). The distance a is the Lagrangian coordinate (Gibson *et al.*, 1981).

The clay layer in the configuration shown by Fig. 1(b) will have a new configuration, as shown in Fig. 1(c), with the progression of consolidation. The datum plane remains fixed. The top surface has moved and the sample deforms to a new position (ABCD). A new distance ξ locates a material point as a function of time; the distance ξ is the convective coordinate (Gibson *et al.*, 1981).

The Lagrangian coordinate a and time t are independent variables, while the convective coordinate ξ is a variable that depends on a and t, and also with assumption (6) that ξ is a function of (a, t). The relationship between a and ξ is given by Gibson *et al.* (1981) as follows

$$\frac{\partial \xi}{\partial a} = \frac{1+e}{1+e_0} \tag{1}$$

in which e is the void ratio of the clay layer, and e_0 is the initial void ratio as for $t = 0$. Thus, at any time t while the top surface (upper boundary) and the bottom (lower boundary) of the layer of clay can be denoted by $a = H$ and $a = 0$, in the convective coordinate system the same boundaries are located at $\xi = \xi_0(t) = H - s_0(t)$ – where $s_0(t)$ is the displacement of the ground surface – and $\xi = 0$, respectively.

General equations in convective coordinate

(*a*) Assuming that the soil particles and the pore water are incompressible, the equations for large-strain radial consolidation of saturated soil in the convective vertical coordinate system are as follows

$$\frac{\partial \sigma(\xi, r, t)}{\partial \xi} = -\frac{(G_s + e)\gamma_w}{1+e} \tag{2a}$$

where σ is the total stress $\sigma = (\xi, r, t)$, G_s is the specific gravity of the solid particles and γ_w is the unit weight of water

In the radial direction

$$\frac{\partial \sigma(\xi, r, t)}{\partial r} = 0 \tag{2b}$$

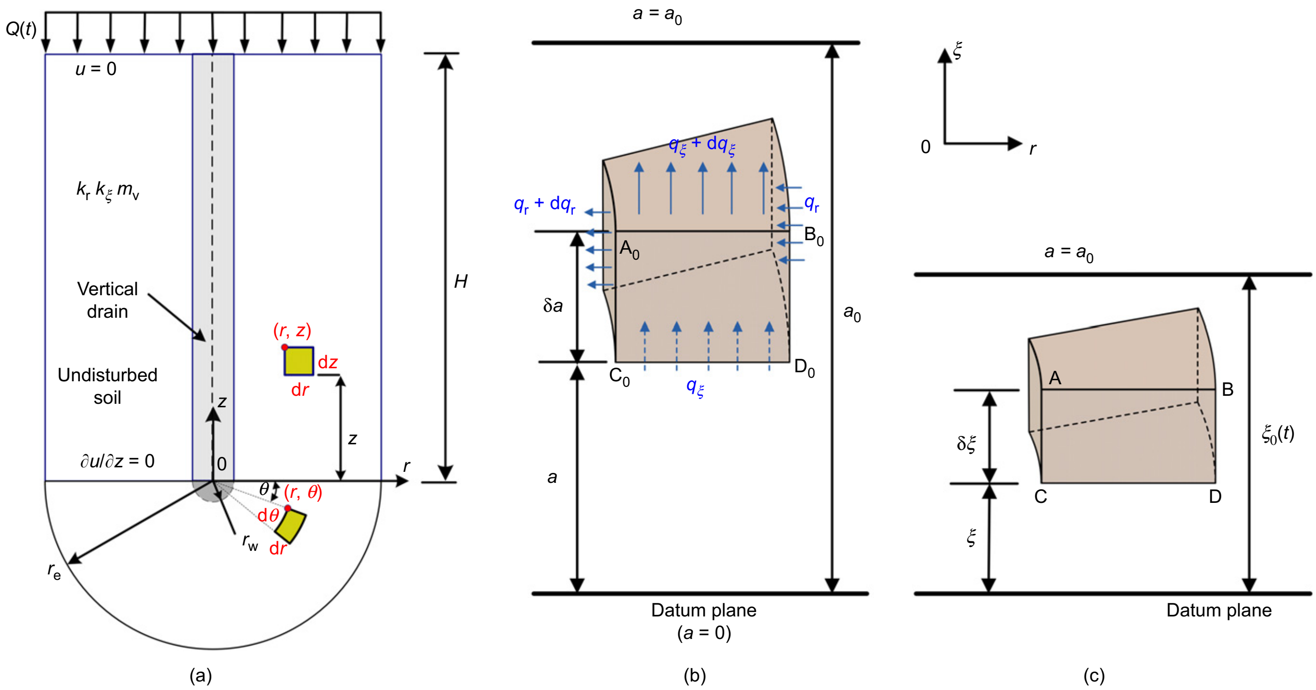

Fig. 1. Coordinate systems: (a) Eulerian coordinate system; (b) convective and Lagrangian coordinate system: initial configuration $t = 0$; (c) convective and Lagrangian coordinate system: current configuration at time t

(*b*) Equation for equilibrium of the pore water

The definition of excess pore pressure is $u = u_t - u_0^*$, where u_t is the total pore-water pressure, u is the excess pore-water pressure and u_0^* is the initial hydrostatic pore-water pressure. Specifically, the initial hydrostatic pore-water pressure is a 'virtual' and time-dependent hydrostatic distribution, with the piezometric level located at the bottom layer in the current configuration as $u_0^*\ (a, t) = \gamma_w(\xi_0 - \xi)$.

Therefore, in the vertical direction

$$\frac{\partial u_t}{\partial \xi} = \frac{\partial u}{\partial \xi} - \gamma_w \tag{3a}$$

and in the radial direction

$$\frac{\partial u_t}{\partial r} = \frac{\partial u}{\partial r} \tag{3b}$$

(*c*) Compressibility equation of effective stress with void ratio

$$e = e_0 - C_c \log_{10}\left(\frac{\sigma'}{\sigma_0'}\right) \tag{4}$$

where e_0 is the initial void ration, C_c is the compression index, σ' is the effective stress and σ_0' is the initial effective stress.

Darcy's law and solid skeleton compressibility equation

(*a*) Darcy's flow

The hydraulic gradients in the vertical direction can be given as

$$i_\xi = \frac{1}{\gamma_w}\frac{\partial u}{\partial \xi} \tag{5a}$$

$$v_\xi = \frac{e}{1+e}(v_\xi^w - v_s) = -k_\xi i_\xi \tag{6a}$$

v_ξ is the apparent velocity of flow in the vertical direction; v_ξ^w is the actual velocity of water in the vertical direction; and v_s is the actual velocity of solid in the vertical direction.

The hydraulic gradient in a radial direction is

$$i_r = \frac{1}{\gamma_w}\frac{\partial u}{\partial r} \tag{5b}$$

$$v_r = \frac{e}{1+e}v_r^w = -k_r i_r \tag{6b}$$

v_r is the apparent velocity of flow in radial direction; v_r^w is the actual velocity of water in the radial direction. Solid only moves in the vertical direction, therefore, there is no actual velocity of solid in the radial direction.

(*b*) Continuity equation

Consider the element shown as Figs 1(a) and 1(b). The element exists at a depth ξ above the bottom of the compressible layer; it has a thickness $d\xi$ and volume $r d\theta dr d\xi$. Volumetric change of the element is the difference between the amount of flow into and out of the element.

Volume changes within the soil element are equal to the net decrease in volume of water, therefore

$$dV = dq_r + dq_\xi \tag{7}$$

in which dV is the volume change of the soil element, dq_r is the volume change in the radial direction and dq_ξ is the volume change in the radial direction.

The quantity of flow

$$q_\xi = \frac{e}{1+e}(v_\xi^w - v_s) r dr d\theta \tag{8}$$

$$q_\xi + dq_\xi = \left\{\frac{e}{1+e}(v_\xi^w - v_s) + \frac{\partial}{\partial \xi}\left[\frac{e}{1+e}(v_\xi^w - v_s)\right] d\xi\right\} r dr d\theta \tag{9}$$

Then

$$dq_\xi = \frac{\partial}{\partial \xi}\left[\frac{e}{1+e}(v_\xi^w - v_s)\right] d\xi r dr d\theta \tag{10}$$

In the radial direction

$$q_r = \frac{e}{1+e} v_r^w r d\xi d\theta \tag{11}$$

$$q_r + dq_r = \left\{\left(\frac{e}{1+e}\right) v_r^w + \frac{\partial}{\partial r}\left[\left(\frac{e}{1+e}\right) v_r^w\right] dr\right\}(r + dr) d\theta d\xi \tag{12}$$

Then

$$dq_r = \left(\frac{e}{1+e}\right) v_r^w d\theta dr d\xi + \frac{\partial}{\partial r}\left[\left(\frac{e}{1+e}\right) v_r^w\right] r d\theta dr d\xi \tag{13}$$

Therefore, the equation for the continuity of pore-water flow is

$$\frac{v_r^w}{r}\frac{e}{1+e} + \frac{\partial}{\partial r}\left(\frac{e}{1+e}v_r^w\right) + \frac{\partial}{\partial \xi}\left[\frac{e}{1+e}\left(v_\xi^w - v_s\right)\right] = -\frac{1}{1+e}\frac{\partial e}{\partial t} \tag{14}$$

An alternative way of deriving equation (14) from a general formulation of the balance of mass equations for solid and water, as suggested by an anonymous reviewer of this technical note, is also included in Appendix 1.

Corresponding Langrangian equations can be obtained by combining equations (1)–(3)

$$\frac{\partial \sigma}{\partial a} = -\frac{(G_s + e)\gamma_w}{1 + e_0} \tag{15}$$

in which σ is the total stress in Langrangian coordinate

$$\frac{\partial u_t}{\partial a} = \frac{\partial u}{\partial a} - \frac{(1+e)\gamma_w}{1+e_0} \tag{16}$$

$$v_\xi^w - v_s = -\frac{k_\xi(1+e_0)}{\gamma_w e}\frac{\partial u}{\partial a} \tag{17}$$

The relationship between the radial component of Darcy's velocity and the radial component of the hydraulic gradient is required

$$v_r^w = -\frac{k_r(1+e)}{\gamma_w e}\frac{\partial u}{\partial r} \tag{18}$$

$$\frac{1}{r\gamma_w}\frac{\partial}{\partial r}\left(r k_r \frac{\partial u}{\partial r}\right) + \frac{1+e_0}{(1+e)\gamma_w}\frac{\partial}{\partial a}\left[k_\xi \frac{\partial u}{\partial a}\frac{1+e_0}{(1+e)}\right] = \frac{1}{1+e}\frac{\partial e}{\partial t} \tag{19}$$

In Gibson's large-strain consolidation theory, simplified assumptions were made in order to reduce the governing equation from a highly non-linear form to a linear form (Gibson *et al.*, 1981; Cai & Geng, 2009)

$$g(e) = -\frac{k_v}{\gamma_w(1+e)}\frac{\partial\sigma'}{\partial e} = \frac{c_v}{(1+e)^2} \tag{20a}$$

and

$$\lambda(e) = -\frac{d}{de}\left(\frac{de}{d\sigma'}\right) \tag{20b}$$

where k_v is the vertical permeability of the soil; c_v is the small-strain coefficient of consolidation; $g(e)$ is the finite-strain coefficient of consolidation; and $\lambda(e)$ is the linearisation constant.

However, the vertical permeability of the soil (k_ξ) is often derived empirically to be an exponential function with base 10 of the void ratio

$$k_\xi = k_{\xi 0} \times 10^{e-e_0/C_{k\xi}} \tag{21}$$

The constitutive laws linking the void ratio e to the radial permeability coefficient k_r and the effective stress σ' use the similarity exponential function, which is as follows (Fig. 2)

$$k_r = k_{r0} \times 10^{e-e_0/C_{kr}} \tag{22}$$

where $C_{k\xi}$ is a vertical permeability change index, the slope of the e–log $_{10}k_\xi$ relationship, and C_{kr} is a radial permeability change index, the slope of the e–log $_{10}k_r$ relationship.

Although equation (20a) gives a good approximation of equation (21) for a given range of void ratios for appropriate types of soil (i.e. soils with C_{kr} values within an appropriate range) (Fig. 3), it is still better to use the one, equation (21), which is more general. This is illustrated in Fig. 3 using results from Tavenas *et al.* (1983), for a plot of e values ranging from 0·1 to 1·8 and $C_{kr} = 3$ (corresponding to Swedish clays) where good agreement between equations (20a) and (21) is evident.

Replacing k_ξ and k_h in equation (19) by equations (21), (22) and (4) and also by considering the principle of effective stress ($\sigma = \sigma' + u_t$), the following equation governing excess pore-water pressure in the Lagrangian system can be given as (note that the detail of derivation can be found in Appendix 2)

$$\begin{aligned}
&\frac{A(1+e)k_{r0}}{r(1+e_0)}\frac{\partial}{\partial r}\left[r10^{(e-e_0)(1/C_{kr}-1/C_c)}\frac{\partial e}{\partial r}\right] \\
&\quad + (1-G_s)k_{\xi 0}\frac{d}{de}\left(\frac{10^{e-e_0/C_{kr}}}{1+e}\right)\frac{\partial e}{\partial a} \\
&\quad - Ak_{\xi 0}\frac{\partial}{\partial a}\left[\frac{(1+e_0)}{(1+e)}10^{(e-e_0)(1/C_{kr}-1/C_c)}\frac{\partial e}{\partial a}\right] \\
&= \frac{1}{1+e_0}\frac{\partial e}{\partial t}
\end{aligned} \tag{23}$$

where $A = \sigma_0' \times \ln 10/\gamma_w C_c$.

BOUNDARY AND INITIAL CONDITIONS

Here the case is considered where the top surface ($a = H$) of the thick layer of clay is pervious but the bottom ($a = 0$) is impervious. Therefore, the boundary conditions in the Langrangian coordinates are

$$\begin{cases}
e(r,H,t) = e_0 - C_c \log_{10}\left[\dfrac{\sigma_0' + Q(t) - p(t)}{\sigma_0'}\right], & \left.\dfrac{\partial e}{\partial a}\right|_{a=0} = \dfrac{(G_s-1)\gamma_w}{1+e_0}\dfrac{C_c}{\sigma_0' \times \ln 10}10^{e-e_0/C_c} \\
e(r_w,a,t) = e_0 - C_c \log_{10}\left[\dfrac{\sigma_0' + Q(t) - p(t) + (G_s-1)\gamma_w a/(1+e_0)}{\sigma_0'}\right], & \left.\dfrac{\partial e}{\partial r}\right|_{r=r_e} = 0
\end{cases} \tag{24}$$

where $Q(t)$ is the time-dependent surcharge loading and $p(t)$ is the time-dependent vacuum loading.

The initial condition corresponding to constant loading can then be expressed as

$$t = 0\; e = e_0 - C_c \log_{10}\left[1 + \frac{(G_s-1)\gamma_w a}{\sigma_0'(1+e_0)}\right] \tag{25}$$

NUMERICAL RESULTS

Equation (23) is highly non-linear and does not have a general solution for the boundary conditions mentioned previously. Therefore, the finite-element solver FlexPDE (Professional Version 6.35; PDE Solutions, 2012), with an automatic adaptive mesh approach, was used to solve equation (23). The two-dimensional numerical finite-element configurations are shown in Fig. 4. The parameters used for this analysis are: $r_w = 0{\cdot}07$ m, $r_e = 0{\cdot}7$ m, $H = 10$ m, $C_c = 0{\cdot}4$, $e_0 = 1{\cdot}1$, $\sigma_0' = 20$ kPa, $k_{r0} = 4{\cdot}48 \times 10^{-4}$ m/day, $k_{\xi 0} = 2{\cdot}27 \times 10^{-4}$ m/day, analysis point A: $r_A = 0{\cdot}68$ m, $H_A = 0{\cdot}3$ m. The rates of $C_c/C_{k\xi}$ and C_c/C_{kr} for soil in the range of 0·5–1·5 are used in the analysis (Berry & Poskitt, 1969; Gibson *et al.*, 1990; Geng *et al.*, 2006). The ranges of specific gravity (G_s) of soils are given in Table 1. According to Table 1, $G_s = 2{\cdot}9$ is the upper limit for a clay and silty clay type of soil. In fact, for a soil with mica or iron, the specific gravity G_s could reach 3·0. For organic soils, the specific gravity will be below 2·0, therefore, $G_s = 1{\cdot}8$ was chosen to represent this type of soil. Furthermore, when $G_s = 1{\cdot}0$, according to the governing equation (23) and the boundary conditions, and the initial condition equations (24) and (25), the large-strain theory should reduce to the small-strain theory. Therefore, in this paper, $G_s = 1$, 1·8 and 2·9 were chosen for the numerical calculations.

The degree of consolidation can be defined either in terms of effective stress, which shows the rate of the increase of effective stress or the rate of dissipation of excess pore-water pressure (U_p), or in terms of settlement, which indicates the rate of settlement development (U_s) (Geng *et al.*, 2006).

The degree of consolidation of the soil defined in terms of effective stress U_p, can be derived as

$$U_p = \frac{\int_0^H \int_{r_w}^{r_e} r\sigma' \mathrm{d}r\mathrm{d}a}{\int_0^H \int_{r_w}^{r_e} r\sigma_f' \mathrm{d}r\mathrm{d}a} \tag{26}$$

where σ_f' is the final effective stress.

The degree of consolidation of the soil defined in terms of strain, U_s, can be derived as

$$U_s = \frac{\int_0^H \int_{r_w}^{r_e} \varepsilon r \mathrm{d}r\mathrm{d}a}{\int_0^H \int_{r_w}^{r_e} \varepsilon_u r \mathrm{d}r\mathrm{d}a} = \frac{\int_0^H \int_{r_w}^{r_e} (e_0 - e) r \mathrm{d}r\mathrm{d}a}{\int_0^H \int_{r_w}^{r_e} (e_0 - e_u) r \mathrm{d}r\mathrm{d}a} \tag{27}$$

where e_u is the final void ratio.

Figure 5 shows a comparison between results for large-strain radial consolidation theory and the classic small-strain radial consolidation theory (Barron, 1948) for the normalised pore-water pressure (u/Q_u) at point A ($r_A = 0{\cdot}68$ m, $H_A = 3{\cdot}0$ m, as shown in Fig. 4). The dissipation of the pore-water pressure is highly affected by the non-linear change of compressibility and the permeability, as well as the self-weight of the soil. There are negligible differences between the large-strain radial theory

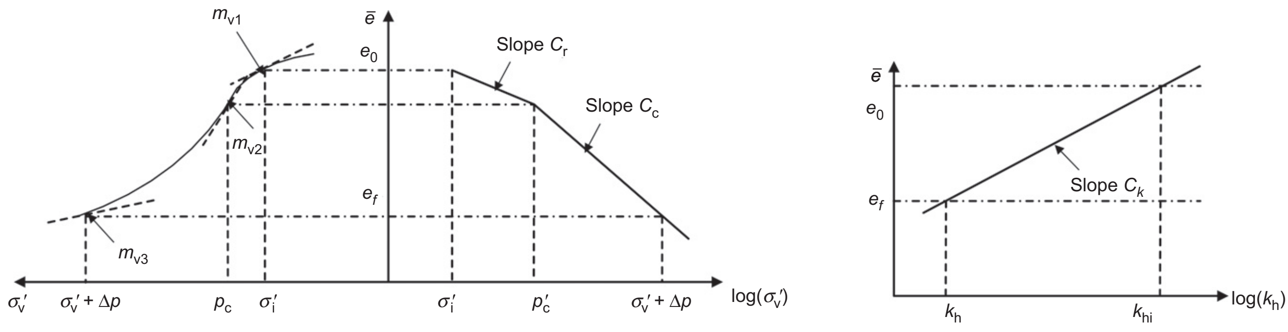

Fig. 2. e–log $_{10}k$ and e–log $_{10}\sigma'$ relationships

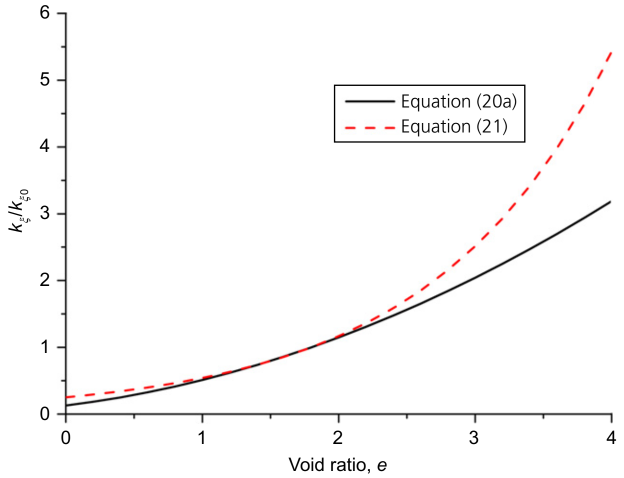

Fig. 3. Comparison of k_ξ–e relationships from equations (20a) and (21) ($e_0 = 1{\cdot}8$; $C_{k\xi} = 3$)

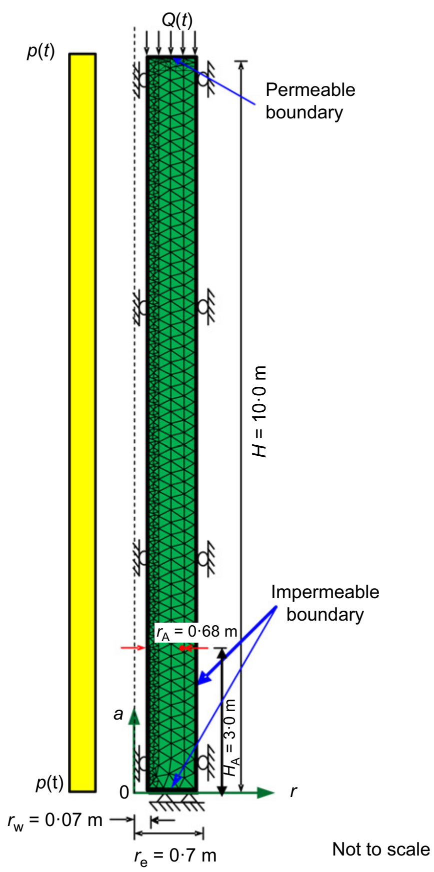

Fig. 4. Finite-element configuration

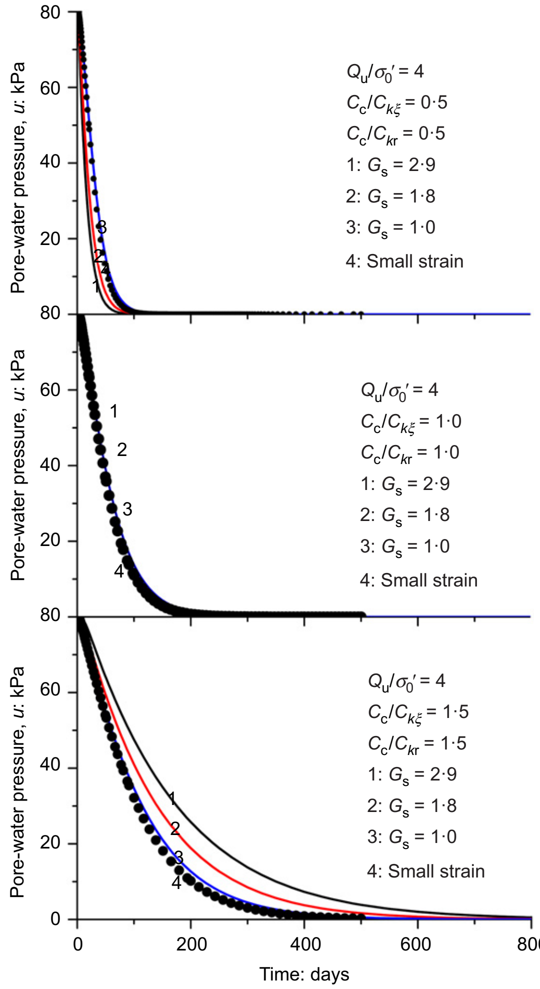

Fig. 5. Comparison of the dissipation of pore-water pressure between non-linear large-strain radial consolidation theory and small-strain radial consolidation theory

and the small-strain radial theory when the compressibility of the soil divided by the permeability of the soil (both radial permeability and vertical permeability) is equal to 1·0 ($C_c/C_{k\xi} = 1$; $C_c/C_{kr} = 1$). When compressibility of the soil over the permeability of the soil (horizontal permeability and vertical permeability) is less than one ($C_c/C_{k\xi} < 1$; $C_c/C_{kr} < 1$), pore-water pressure dissipates faster with the consideration of self-weight of the soil compared to classical small-strain theory. In contrast, when compressibility of the soil over the permeability of the soil (horizontal permeability and vertical permeability) is greater than one ($C_c/C_{k\xi} > 1$; $C_c/C_{kr} > 1$), the actual pore-water pressure dissipates more slowly with the consideration of self-weight of the soil

Table 1. Soil specific gravities

Soil	Specific gravity
Sand	2·63–2·67
Silt	2·65–2·7
Clay and silty clay	2·67–2·9
Organic soils	<2·0

compared to the classical small-strain theory. In other words, from $e \sim \log_{10}\sigma'$ and $e \sim \log_{10}k$ relationships (equations (20)–(22)), it can be observed that the void radio decreased with the increase of the effective stress (σ') as well as the decrease in the permeability (k_r and k_ξ). The void ratio will not be affected by the changes of the effective stress and the permeability only for the case when the ratio of the increase in effective stress (σ') and the decrease in the permeability (k_r and k_ξ) equals 1 during the consolidation process, and this is very unlikely to happen in most cases. Therefore, it is necessary to consider the influence of the self-weight of the soil by using large-strain radial consolidation. Moreover, the actual dissipation of pore-water pressure is faster or slower than the results obtained from the classic radial consolidation theory depending on which factor is more dominant in relation to the decrease of the void ratio, the increase in the effective stress (σ') or the decrease in the permeability (k_r and k_ξ). If the changes of effective stress have less influence than changes in the permeability on the decrease of the void ratio ($C_c/C_{k\xi} < 1$; $C_c/C_{kr} < 1$), classic small-strain radial consolidation theory will underestimate the dissipation of the pore-water pressure. If the changes of effective stress have more influence than changes in the permeability on the decrease of the void ratio ($C_c/C_{k\xi} > 1$; $C_c/C_{kr} > 1$), classic small-strain radial consolidation theory will overestimate the dissipation of the pore-water pressure. Therefore, in order to evaluate the actual consolidation process more accurately, variations in permeability and compressibility, self-weight of the soil, stress history and the magnitude of preloading pressure should all be considered when running the consolidation analysis.

Figure 6 shows the influence of the load increment ratio (Q_u/σ'_0) on different degrees of consolidation for large-strain theory (U_p and U_s). When $C_c/C_{k\xi} < 1$ and $C_c/C_{kr} < 1$, the increase of the load increment ratio (Q_u/σ'_0) has negligible influence on U_p. The increase of the load increment ratio (Q_u/σ'_0) will influence U_s more compared to the influence on U_p, especially in the middle of the consolidation process. However, U_s remains almost the same with the changing of the Q_u/σ'_0 when $C_c/C_{k\xi} > 1$ and $C_c/C_{kr} > 1$. The degree of consolidation defined by pore-water pressure (U_p) decreases with the increase of the load increment ratio when $C_c/C_{k\xi} \geq 1$ and $C_c/C_{kr} \geq 1$.

The influences of individual compressibility over permeability ratio ($C_c/C_{k\xi}$ and C_c/C_{kr}) on the degree of

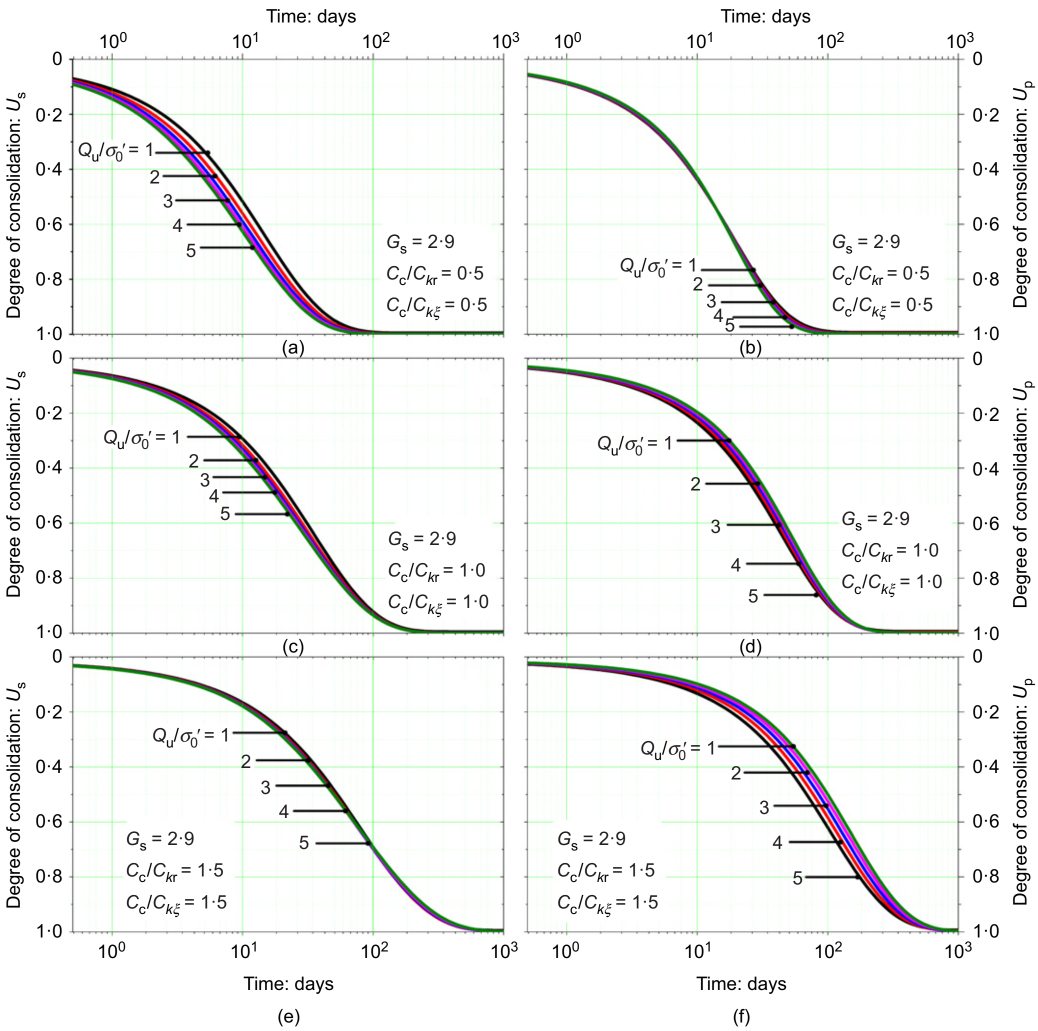

Fig. 6. Degree of consolidation U (U_s and U_p) plotted against time for varying load increment ratio (Q_u/σ'_0): (a) U_s for $C_c/C_{kr} = 0{\cdot}5$, $C_c/C_{k\xi} = 0{\cdot}5$; (b) U_p for $C_c/C_{kr} = 0{\cdot}5$, $C_c/C_{k\xi} = 0{\cdot}5$; (c) U_s for $C_c/C_{kr} = 1{\cdot}0$, $C_c/C_{k\xi} = 1{\cdot}0$; (d) U_p for $C_c/C_{kr} = 1{\cdot}0$, $C_c/C_{k\xi} = 1{\cdot}0$; (e) U_s for $C_c/C_{kr} = 1{\cdot}5$, $C_c/C_{k\xi} = 1{\cdot}5$; (f) U_p for $C_c/C_{kr} = 1{\cdot}5$, $C_c/C_{k\xi} = 1{\cdot}5$

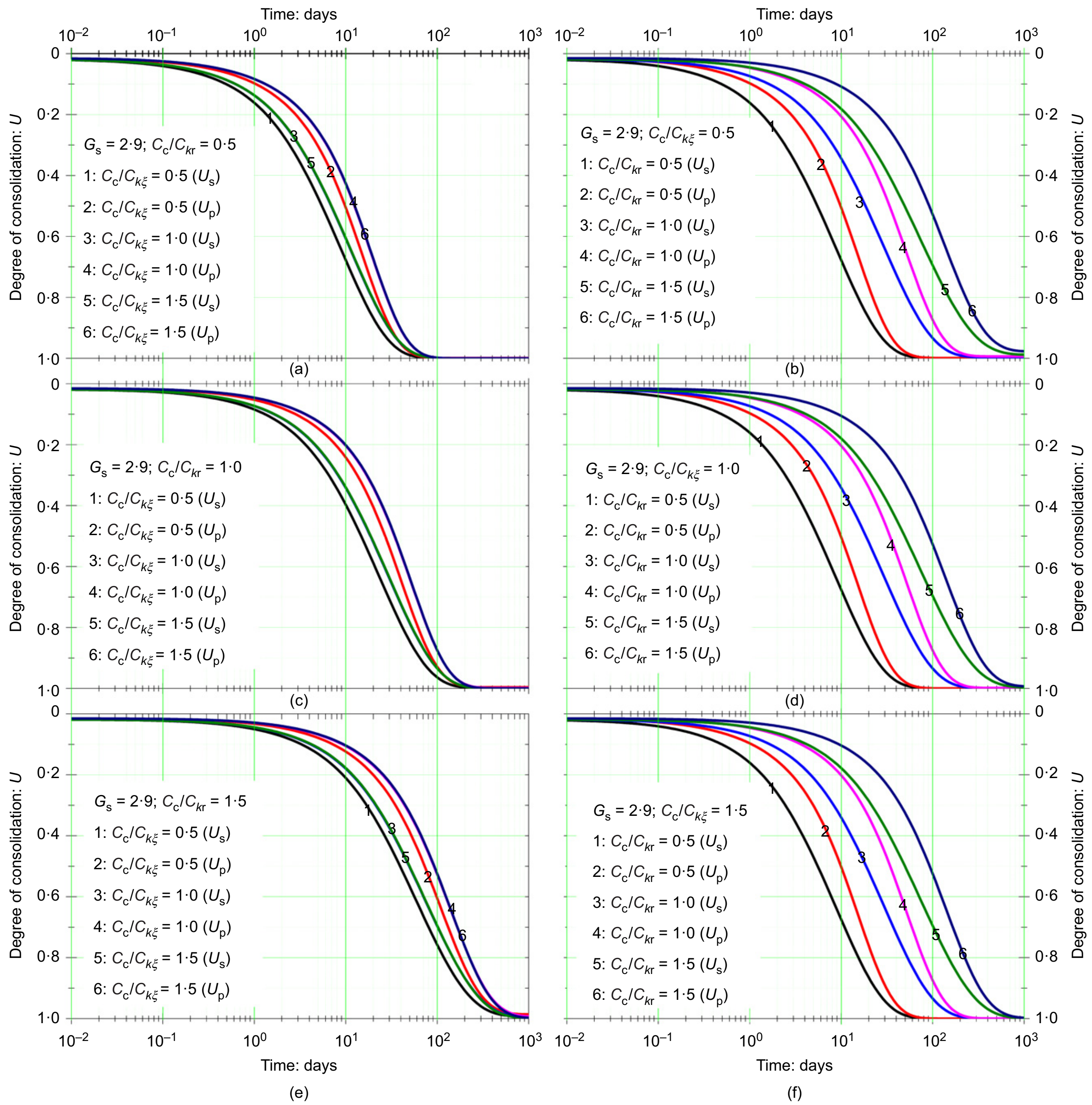

Fig. 7. Changes in the degree of consolidation varying with compressibility and permeability ratio: (a) $C_c/C_{kr} = 0{\cdot}5$; (b) $C_c/C_{k\xi} = 0{\cdot}5$; (c) $C_c/C_{kr} = 1{\cdot}0$; (d) $C_c/C_{k\xi} = 1{\cdot}0$; (e) $C_c/C_{kr} = 1{\cdot}5$; (f) $C_c/C_{k\xi} = 1{\cdot}5$

consolidation (U_p and U_s) are shown in Fig. 7. Comparing $C_c/C_{k\xi}$ to the ratio C_c/C_{kr}, the latter has a greater influence on the degree of consolidation, which means horizontal permeability has a major influence on the consolidation progress, rather than the vertical permeability, in a vertical drain system. With the same C_c/C_{kr} ratio, the influence of $C_c/C_{k\xi}$ could be neglected when $C_c/C_{k\xi} \geq 1$. At the same time (t), U_p is always less than U_s, which is similar to the one-dimensional, non-linear, small-strain consolidation obtained by Geng *et al.* (2006), Geng (2008) and Cai & Geng (2009). This also shows that the degree of consolidation calculated based on U_s occurs at a slightly higher rate than the degree of consolidation calculated based on U_p.

CONCLUSIONS

Based on Gibson's large-strain theory and Barron's free-strain theory, a constitutive model has been presented that incorporates the radial and vertical flows, weight of the soil (G_s), vacuum preloading and the variable hydraulic conductivity (k_r and k_ξ), compressibility index (C_c) and permeability change index ($C_{k\xi}$ and C_{kr}) during consolidation. The $e \sim \log_{10} k_r$, $e \sim \log_{10} k_\xi$ and $e \sim \log_{10} \sigma'$ relationship were used to represent the non-linear relationship of the soil. By using the finite-element method, some results could be given, and the difference between large-strain radial consolidation theory and the small-strain theory was found to depend on the variable hydraulic conductivity, compressibility and the weight of the soil, as well as the value of external loading. When $C_c/C_{kr} = 1$ and $C_c/C_{k\xi} = 1$, the actual rate of consolidation was the same as the classic small-strain theory (Barron's theory). When $C_c/C_{kr} \geq 1$, the actual dispersion of pore-water pressure process calculated by large-strain theory took place at a slower rate than the conventional solution. Moreover, the rate of consolidation defined by the dissipation of pore-water pressure (U_p) was found to decrease with the increase of the Q_u/σ_0' values. However, when $C_c/C_{kr} \geq 1$, according to the results from large-strain theory, the increase of the Q_u/σ_0' values did not have much influence on the settlement or U_s, which also shows that the settlement occurs faster than excess pore pressure dissipation. Also, with the same C_c/C_{kr} ratio, the influence of $C_c/C_{k\xi}$ could be neglected when $C_c/C_{k\xi} \geq 1$.

APPENDIX 1. ALTERNATIVE DERIVATION OF EQUATION (14)

Working in direct (index-free) notation, for a general deformation and flow process, the balance of mass of incompressible solid and fluid phases require that

$$-\frac{\partial n}{\partial t} + \nabla \cdot [(1-n)\boldsymbol{v}^{\mathrm{s}}] = 0$$

$$\frac{\partial n}{\partial t} + \nabla \cdot [n\boldsymbol{v}^{\mathrm{w}}] = 0$$

where n is the porosity of the soil, see Coussy (2004: p. 12, equations (1.59a,b)). Combined together they provide

$$\nabla \cdot [n(\boldsymbol{v}^{\mathrm{w}} - \boldsymbol{v}^{\mathrm{s}})] + \nabla \cdot \boldsymbol{v}^{\mathrm{s}} = 0 \tag{28}$$

Considering that the divergence of solid skeleton velocity is linked to the material time derivative of the Jacobian of the deformation gradient $\boldsymbol{F}$

$$J = \det(\boldsymbol{F}) \quad \text{with} \quad \boldsymbol{F} = \frac{\partial \boldsymbol{\xi}(\boldsymbol{X}, t)}{\partial \boldsymbol{X}}$$

by the relation (see Marsden & Hughes, 1983: p. 86)

$$\nabla \cdot \boldsymbol{v}^{\mathrm{s}} = \frac{\dot{J}}{J}$$

where a superposed dot indicates a material time derivative, equation (28) transforms into

$$\nabla \cdot [n(\boldsymbol{v}^{\mathrm{w}} - \boldsymbol{v}^{\mathrm{s}})] = -\frac{\dot{J}}{J} \tag{29}$$

In the present case of one-dimensional deformation, in which

$$\boldsymbol{X} = \begin{Bmatrix} a \\ R \end{Bmatrix} \quad \boldsymbol{\xi} = \begin{Bmatrix} \xi(a,t) \\ r(R) \end{Bmatrix} = \begin{Bmatrix} \xi(a,t) \\ R \end{Bmatrix}$$

$$\boldsymbol{F} = \begin{bmatrix} \partial\xi/\partial a & 0 \\ 0 & 1 \end{bmatrix} \quad J = \frac{\partial \xi}{\partial a} = \frac{1+e}{1+e_0}$$

Equation (29) yields

$$\nabla \cdot [n(\boldsymbol{v}^{\mathrm{w}} - \boldsymbol{v}^{\mathrm{s}})] = \nabla \cdot \left[\frac{e}{1+e}(\boldsymbol{v}^{\mathrm{w}} - \boldsymbol{v}^{\mathrm{s}})\right] = -\frac{\dot{e}}{1+e} \tag{30}$$

Finally, remembering that in cylindrical coordinates the (spatial) divergence of a vector $\boldsymbol{V}$ can be expressed as (Malvern, 1969: p. 667)

$$\nabla \cdot \boldsymbol{V} = \frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_\xi}{\partial \xi}$$

and noting that the problem is axially symmetric and $v_r^{\mathrm{s}} = 0$, equation (30) finally yields

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{e}{1+e}v_r^{\mathrm{w}}\right) + \frac{\partial}{\partial \xi}\left[\frac{e}{1+e}\left(v_\xi^{\mathrm{w}} - v_\xi^{\mathrm{s}}\right)\right] = -\frac{\dot{e}}{1+e} \tag{31}$$

APPENDIX 2: DERIVATION OF THE GOVERNING EQUATION (23)

From equation (4), the following equation can be obtained in the vertical direction

$$\frac{\partial \sigma'}{\partial a} = \frac{\sigma_0' \times \ln 10}{C_c} 10^{e_0 - e/C_c} \frac{\partial e}{\partial a} \tag{32}$$

and in the radial direction

$$\frac{\partial \sigma'}{\partial r} = \frac{\sigma_0' \times \ln 10}{C_c} 10^{e_0 - e/C_c} \frac{\partial e}{\partial r} \tag{33}$$

By considering the principle of effective stress ($\sigma = \sigma' + u_t$), equations (32) and (33) after differentiation can be changed as follows.

In the vertical direction

$$\frac{\partial \sigma}{\partial a} = \frac{\sigma_0' \times \ln 10}{C_c} 10^{e_0 - e/C_c} \frac{\partial e}{\partial a} + \frac{\partial u_t}{\partial a} \tag{34}$$

In the radial direction

$$\frac{\partial \sigma}{\partial r} = \frac{\sigma_0' \times \ln 10}{C_c} 10^{e_0 - e/C_c} \frac{\partial e}{\partial r} + \frac{\partial u_t}{\partial r} \tag{35}$$

Substituting equations (15) and (16) (into) equation (34) yields

$$\frac{\partial u}{\partial a} = -\frac{\sigma_0' \times \ln 10}{C_c} 10^{e_0 - e/C_c} \frac{\partial e}{\partial a} + \frac{(1+e)\gamma_w}{1+e_0} - \frac{(G_s + e)\gamma_w}{1+e_0} \tag{36}$$

By substituting equations (2b) and (3b) (into) equation (35), the following is obtained

$$\frac{\partial u}{\partial r} = \frac{\sigma_0' \times \ln 10}{C_c} 10^{e_0 - e/C_c} \frac{\partial e}{\partial r} \tag{37}$$

Replacing k_ξ and k_h in equation (19) by equations (21) and (22) and substituting equations (36) and (37), (the) governing equation (23) can be obtained.

NOTATION

a	distance, Lagrangian coordinate
C_c	compression index
C_{kr}	radial permeability change index
$C_{k\xi}$	vertical permeability change index
c_v	small-strain coefficient of consolidation
dq_r	volume change in radial direction
dq_ξ	volume change in vertical direction
dV	volume change of soil element
e	void ratio of clay layer
e_u	final void ratio
e_0	initial void ratio of clay layer
$\boldsymbol{F}$	Jacobian of the deformation gradient
G_s	specific gravity of solid particles
$g(e)$	finite-strain coefficient of consolidation
H	height; and initial thickness of saturated clay layer
J	Jacobian of deformation gradient F
k_r	radial permeability coefficient
k_v	vertical permeability of soil
k_ξ	vertical permeability of soil
n	porosity of soil
$p(t)$	time-dependent vacuum loading
$Q(t)$	time-dependent surcharge loading
R	radial distance
r	radial axis
r_e	vertical drain external radius
$s_0(t)$	displacement of ground surface
t	time
U_p	degree of consolidation of soil defined in terms of effective stress
U_s	degree of consolidation of soil defined in terms of strain
u_t	total pore-water pressure
u_0^*	initial hydrostatic pore-water pressure
v_r	apparent velocity of flow in radial direction
v_r^s	radial velocity of solid
v_r^w	actual velocity of water in radial direction
v_s	actual velocity of solid in vertical direction
v_ξ	apparent velocity of flow in vertical direction
v_ξ^w	actual velocity of water in vertical direction
$\boldsymbol{v}^s$	skeleton velocity
$\boldsymbol{v}^w$	fluid velocity
$\boldsymbol{X}$	position vector
γ_w	unit weight of water
$\lambda(e)$	linearisation constant

ξ vertical axes
σ total stress
σ' effective stress
σ'_f final effective stress
σ'_0 initial effective stress

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