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Cole-Hopf–Feynman-Kac formula and quasi-invariance for Navier-Stokes equations

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Abstract. We make a refined comparison between the Navier-Stokes equations and their dynamically-scaled Leray equations solely on the basis of their scaling property. Previously it was observed using the vector potentials that they differ only by one drift term [Ohkitani (2017)]. The Duhamel principle recasts the equations in path integral forms, which differ by *two* Maruyama-Girsanov densities. In this brief paper we simplify the concept of quasi-invariance (or, near-invariance) by combining the result with a Cole-Hopf transform and the Feynman-Kac formula. That way, as a multiplicative characterisation we can place those equations just *one* Maruyama-Girsanov density apart. Furthermore, as an additive characterisation we express the difference in terms of the Malliavin H-derivative.

Keywords: Navier-Stokes equations, Leray equations, dynamic scaling, critical spaces, Cameron-Martin-Maruyama-Girsanov theorem, Malliavin derivative, global regularity

1. Introduction

The theory of the Navier-Stokes equations [13] has a long history, see e.g. [3, 7, 6], or [22] for references cited therein. There have been substantial progress recently, including the results regarding critical norms such as $\|\boldsymbol{u}\|_{L^3}$ or $\|\boldsymbol{u}\|_{\dot{H}^{1/2}}$. These norms are defined respectively by $\|\boldsymbol{u}\|_{L^3}^3 = \int_{\mathbb{R}^3} |\boldsymbol{u}|^3 d\boldsymbol{x}$ and $\|\boldsymbol{u}\|_{\dot{H}^{1/2}}^2 = \int_{\mathbb{R}^3} |\boldsymbol{k}||\widehat{\boldsymbol{u}}(\boldsymbol{k})|^2 d\boldsymbol{k}$, where $\widehat{\boldsymbol{u}}(\boldsymbol{k})$ denotes a Fourier transform of the velocity $\boldsymbol{u}(\boldsymbol{x})$. As a motivation for the current research, we recall some of the known results: i) the regularity criterion using $\|\boldsymbol{u}\|_{L^3}$ [8] and ii) the global regularity result for small initial data; in $\|\boldsymbol{u}\|_{\dot{H}^{1/2}}$ [11] and in $\|\boldsymbol{u}\|_{\mathrm{BMO}^{-1}}$ [12].‡ We recall the standard embedding $\dot{H}^{1/2} \subset L^3 \subset \mathrm{BMO}^{-1}$, which follows from $\|\boldsymbol{u}\|_{\dot{H}^{1/2}} \geq c\|\boldsymbol{u}\|_{L^3} \geq c'\|\boldsymbol{u}\|_{\mathrm{BMO}^{-1}}$ with constants c, c'. Both the L^3 and $\dot{H}^{1/2}$-norms are 'extensive,' defined with spatial integrals, as opposed to 'intensive' norms defined with sup operations, which are local in space.

It seems that intensive norms are more suitable for the analysis of the Navier-Stokes equations because it is expected in incompressible fluids integrations are weakly nonlocal through the action of pressure term [1]. It would then be advantageous to take up dependent variables, which are critical and can be treated as an 'intensive norm'. This can be achieved by using the vector potential in 3D and the stream function in 2D.

One example of benefits of such an approach is the following. (See Section 2 for details.) Consider the Navier-Stokes equations in the velocity variable, which is subcritical, and apply the Duhamel principle to rewrite them as integral equations. The so-called Serrin's regularity condition $\int_0^T \|\boldsymbol{u}\|_{L^\infty}^2 dt < \infty$ is well-known for regularity of solutions on $[0, T]$. However, non-trivial efforts would be required to deduce local existence by successive approximations and identify a condition under which classical solutions can be extended.

In contrast, basically the same criterion is obtained immediately once the equations are written in the vector potentials and the Cole-Hopf transform is introduced [23]. This is because in converting to the path-integral equations the Feynman-Kac theorem states that solutions are smooth provided that the potential term is bounded. In other words, we can identify the boundedness condition for the potential term (to be a martingale) as the Serrin's condition for regularity.

By applying dynamic scaling to the Navier-Stokes equations written in the vector potential, it was found [22] that the Navier-Stokes and the Leray equations are just one drift term apart. Moving onto path-integral forms, it was also found that the equations are identical up to two Maruyama-Girsanov densities G. Our purpose here is to nail down the difference to just one G, by combining the above idea with the Cole-Hopf transform [23], followed by an application of the Feynman-Kac formula.

The rest of the paper is organised as follows. In section 2 we apply the Cole-Hopf

‡ We mean by BMO a class of functions f of bounded mean oscillations, defined with the semi-norm $\|f\|_{\mathrm{BMO}} = \sup_Q \frac{1}{|Q|} \int_Q |f(\boldsymbol{x}) - f_Q| \, d\boldsymbol{x}$, where $f_Q = \frac{1}{|Q|} \int_Q |f(\boldsymbol{x})| \, d\boldsymbol{x}$ denotes the average over a volume $|Q|$ of a cube Q. Note also that $\boldsymbol{u} \in \mathrm{BMO}^{-1} \iff \boldsymbol{u} = \partial\psi,\ \psi \in \mathrm{BMO}$, where ∂ denotes a spatial derivative.

transform to the Navier-Stokes and Leray equations written in the vector potential. We then apply the Feynman-Kac formula to convert them to path-integral forms. In Section 3, by changing probability measures using the Cameron-Martin-Maruyama-Girsanov theorems, we remove or add the effect of the drift term, thereby characterising the difference in both multiplicative and additive manners. Section 4 is devoted to a summary.

2. Basic equations

2.1. Navier-Stokes and Leray equations

We consider the incompressible Navier-Stokes equations with standard notations in $\mathbb{R}^3$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla p + \frac{1}{2}\triangle\boldsymbol{u}, \tag{1}$$

$$\nabla\cdot\boldsymbol{u} = 0,$$

$$\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}).$$

Assuming that a solution to the Navier-Stokes equations blows up at $t = t_*$, we apply the dynamic scaling transformations e.g. [4, 14, 20, 22]

$$\boldsymbol{u}(\boldsymbol{x},t) = \frac{1}{\sqrt{2a(t_* - t)}}\boldsymbol{U}(\boldsymbol{\xi},\tau), \tag{2}$$

$$\boldsymbol{\xi} = \frac{\boldsymbol{x}}{\sqrt{2a(t_* - t)}},\ \tau = \int_0^t \frac{ds}{\lambda(s)^2} = \frac{1}{2a}\log\frac{t_*}{t_* - t}, \tag{3}$$

where $\lambda(t) = \sqrt{2a(t_* - t)}$ and $a > 0$ is a zooming parameter. We then obtain the non-steady version of the Leray equations

$$\frac{\partial \boldsymbol{U}}{\partial \tau} + \boldsymbol{U}\cdot\nabla_{\boldsymbol{\xi}}\boldsymbol{U} + a(\boldsymbol{\xi}\cdot\nabla_{\boldsymbol{\xi}}\boldsymbol{U} + \boldsymbol{U}) = -\nabla_{\boldsymbol{\xi}}P + \frac{1}{2}\triangle_{\boldsymbol{\xi}}\boldsymbol{U}, \tag{4}$$

$$\nabla_{\boldsymbol{\xi}}\cdot\boldsymbol{U} = 0.$$

We take $2at_* = 1$ so as to make the initial condition in common $\boldsymbol{U}(\cdot,0) = \boldsymbol{u}_0(\cdot)$. The equations (1) and (4) differ by the two terms multiplied by the factor a, which represent drift and damping. Hence the Navier-Stokes equations written in $\boldsymbol{u}$ are not invariant under dynamic scaling transform. It is known [16] that under mild conditions there are no nontrivial steady solutions to (4).

We introduce the vector potentials $\boldsymbol{\psi}$ defined in such a way that $\boldsymbol{u} = \nabla\times\boldsymbol{\psi}$ and $\nabla\cdot\boldsymbol{\psi} = 0$. The Navier-Stokes equations can then be written as a nonlocal version of the Hamilton-Jacobi equations [19]

$$\frac{\partial \boldsymbol{\psi}}{\partial t} - \frac{1}{2}\triangle\boldsymbol{\psi} = \boldsymbol{T}[\nabla\boldsymbol{\psi}], \tag{5}$$

where

$$\boldsymbol{T}[\nabla\boldsymbol{\psi}] \equiv \frac{3}{4\pi}\fint_{\mathbb{R}^3} \frac{[\boldsymbol{r}\cdot(\nabla\times\boldsymbol{\psi}(\boldsymbol{y}))]\,\boldsymbol{r}\times(\nabla\times\boldsymbol{\psi}(\boldsymbol{y}))}{|\boldsymbol{r}|^5}\,\mathrm{d}\boldsymbol{y}, \tag{6}$$

with $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$ and $\fint$ denotes a principal-value integral. We assume that $|\boldsymbol{\psi}(\boldsymbol{x},t)| \to 0$ as $|\boldsymbol{x}| \to \infty$ for all $t \geq 0$. Note that $\nabla\cdot\boldsymbol{T}[\nabla\boldsymbol{\psi}] = 0$ is satisfied. (See [18] for the 2D counterpart.)

The dynamic scaling transformation for $\boldsymbol{\psi}(\boldsymbol{x},t)$ is defined by [22]

$$\boldsymbol{\psi}(\boldsymbol{x},t) = \boldsymbol{\Psi}(\boldsymbol{\xi},\tau) \tag{7}$$

and it satisfies

$$\frac{\partial\boldsymbol{\Psi}}{\partial\tau} - \frac{1}{2}\triangle_{\boldsymbol{\xi}}\boldsymbol{\Psi} + a\boldsymbol{\xi}\cdot\nabla_{\boldsymbol{\xi}}\boldsymbol{\Psi} = \frac{3}{4\pi}\fint_{\mathbb{R}^3} \frac{\boldsymbol{\rho}\times(\nabla\times\boldsymbol{\Psi}(\boldsymbol{\xi}'))\,\boldsymbol{\rho}\cdot(\nabla\times\boldsymbol{\Psi}(\boldsymbol{\xi}'))}{|\boldsymbol{\rho}|^5}\,\mathrm{d}\boldsymbol{\xi}', \tag{8}$$

where $\boldsymbol{\rho} = \boldsymbol{\xi} - \boldsymbol{\xi}'$ and $\boldsymbol{\psi}(\cdot,0) = \boldsymbol{\Psi}(\cdot,0)$. The difference between (5) and (8) is just one drift term, which is minimal due to the critical nature of $\boldsymbol{\psi}$.

2.2. Cole-Hopf and Feynman-Kac formulas

We introduce the Cole-Hopf transform[23] by

$$\psi_j = k\log\theta_j, \quad (j = 1,2,3), \tag{9}$$

with a constant $k(\neq 0)$ and derive the equations for $\boldsymbol{\theta}$. Their derivation is straightforward and brief, but best stated here for completeness

$$\frac{\partial\psi_j}{\partial t} - T_j[\nabla\boldsymbol{\psi}] - \frac{1}{2}\triangle\psi_j = \frac{k}{\theta_j}\frac{\partial\theta_j}{\partial t} - k^2 T_j\left[\frac{\nabla\theta_1}{\theta_1},\frac{\nabla\theta_2}{\theta_2},\frac{\nabla\theta_3}{\theta_3}\right] - \frac{1}{2}k\left(\frac{\triangle\theta_j}{\theta_j} - \frac{|\nabla\theta_j|^2}{\theta_j^2}\right)$$

$$= k\left\{\frac{1}{\theta_j}\left(\frac{\partial\theta_j}{\partial t} - \frac{1}{2}\triangle\theta_j\right) - \left(k\,T_j\left[\frac{\nabla\theta_1}{\theta_1},\frac{\nabla\theta_2}{\theta_2},\frac{\nabla\theta_3}{\theta_3}\right] - \frac{1}{2}\frac{|\nabla\theta_j|^2}{\theta_j^2}\right)\right\},$$

where no summation over j is implied. Setting the right-hand side to zero, we obtain a system of heat equations with a potential term

$$\frac{\partial\theta_j}{\partial t} = \frac{1}{2}\triangle\theta_j + f_j(\boldsymbol{x},t)\theta_j, \quad \text{(no summation)} \tag{10}$$

where

$$f_j(\boldsymbol{x},t) \equiv kT_j\left[\frac{\nabla\theta_1}{\theta_1},\frac{\nabla\theta_2}{\theta_2},\frac{\nabla\theta_3}{\theta_3}\right] - \frac{1}{2}\frac{|\nabla\theta_j|^2}{\theta_j^2}, \quad j = 1,2,3. \text{ (no summation)}$$

A more specific form of $\boldsymbol{T}[\nabla\boldsymbol{\psi}]$ can be obtained by substituting

$$\nabla\times\boldsymbol{\psi} = k\left(\frac{1}{\theta_3}\frac{\partial\theta_3}{\partial x_2} - \frac{1}{\theta_2}\frac{\partial\theta_2}{\partial x_3}, \frac{1}{\theta_1}\frac{\partial\theta_1}{\partial x_3} - \frac{1}{\theta_3}\frac{\partial\theta_3}{\partial x_1}, \frac{1}{\theta_2}\frac{\partial\theta_2}{\partial x_1} - \frac{1}{\theta_1}\frac{\partial\theta_1}{\partial x_2}\right)$$

into (6). Regarding the nonlinear term as forcing in the spirit of Duhamel principle, we rewrite (10) by the Feynman-Kac formula in a form of integral equations

$$\theta_j(\boldsymbol{x},t) = \mathbb{E}\left[\theta_j(\boldsymbol{W}_t,0)\exp\left(\int_0^t f_j(\boldsymbol{W}_s,s)ds\right)\right]. \text{ (no summation)} \tag{11}$$

Here $\boldsymbol{W}_t$ denotes Brownian motion starting from the origin $\boldsymbol{x} = 0$ at $t = 0$, that is, $\boldsymbol{W}_0 = 0$ and $\mathbb{E}$ an expectation value with respect to the corresponding Gaussian probability measure. See **Appendix** for alternative forms of functional integrals. The path-integral expression (11) is a way of writing down the Navier-Stokes equations. We recall that a condition for the regularity $\int_0^t \sup_{\boldsymbol{x}} |f_j(\boldsymbol{x},s)|ds < \infty,\ \ (j = 1,2,3)$ follows from (11). (See also [21].) We note a corresponding formula in two dimensions has been used to capture near-singularities in 2D turbulence [23].

We hereafter use the following notation for convenience

$$F_j[\boldsymbol{\theta}](\boldsymbol{W}_t) \equiv \theta_j(\boldsymbol{W}_t,0)\exp\left(\int_0^t f_j(\boldsymbol{W}_s,s)ds\right). \tag{12}$$

3. Change of probability measures

A method is available in stochastic analysis that allows us to add or remove the effect of drift, the so-called Cameron-Martin-Maruyama-Girsanov theorems [22]. We refer e.g. [2, 15, 17, 24, 25, 26, 28, 29] for stochastic analysis in general. In this section, we make a formal comparison between the two equations using these tools.

$$\begin{array}{ccc}
\text{scale invariance} & & \\
\boldsymbol{\theta}(\boldsymbol{x},t) & = & \boldsymbol{\Theta}(\boldsymbol{\xi},\tau) \\
\text{N-S}\ \Big\| & & \text{Leray}\ \Big\| \\
\mathbb{E}\left[\boldsymbol{F}[\boldsymbol{\theta}](\boldsymbol{W}_t)\right] & & \mathbb{E}\left[\boldsymbol{F}[\boldsymbol{\Theta}](\boldsymbol{W}_\tau + a\boldsymbol{h}(\tau))\right] \\
\text{M-G}\ \Big\| & & \text{C-M}\ \Big\| \\
\mathbb{E}\left[\boldsymbol{F}[\boldsymbol{\theta}](\boldsymbol{W}_t + a\boldsymbol{h}(t))\widehat{G}_a(t)\right] & & \mathbb{E}\left[\boldsymbol{F}[\boldsymbol{\Theta}](\boldsymbol{W}_\tau)G_a(\tau)\right]
\end{array}$$

Figure 1: Scale-invariance, the dynamical equations and the transformation of probability measures; N-S stands for the Navier-Stokes equations, M-G for Maruyama-Girsanov theorem and C-M for Cameron-Martin theorem.

3.1. Leray equations

We consider the Leray equations first, because it has a global smooth solution by construction. Defining $\boldsymbol{\Theta}$ by $\Psi_j = k\log\Theta_j,\ \ (j = 1,2,3)$ the scale-invariance is represented by

$$\boldsymbol{\theta}(\boldsymbol{x},t) = \boldsymbol{\Theta}(\boldsymbol{\xi},\tau).$$

The drift term is taken as $\boldsymbol{b}(\boldsymbol{x}) = -\boldsymbol{x}$ and $\boldsymbol{h}(t) = \int_0^t \boldsymbol{b}(\boldsymbol{W}_s)ds$. We write $(\boldsymbol{x}, t)$ for $(\boldsymbol{\xi}, \tau)$ for a more direct comparison. (See **Figure**.1 for a list of relationships with independent variables distinguished.) The transformed variable $\boldsymbol{\Theta}$ satisfies the following equations

$$\boldsymbol{\Theta} = \mathbb{E}\big[\boldsymbol{F}[\boldsymbol{\Theta}](\boldsymbol{W}_t + a\boldsymbol{h}(t))\big], \ \text{ all } \ t \geq 0 \tag{13}$$

$$= \mathbb{E}\big[\boldsymbol{F}[\boldsymbol{\Theta}](\boldsymbol{W}_t)G_a\big], \ \ 0 \leq t < \frac{\sqrt{2}}{a} \tag{14}$$

where the Cameron-Martin theorem has been applied with the density

$$G_a = \exp\left(a\int_0^t \boldsymbol{b}(\boldsymbol{W}_s)\cdot d\boldsymbol{W}_s - \frac{a^2}{2}\int_0^t |\boldsymbol{b}(\boldsymbol{W}_s)|^2 ds\right).$$

In (13), $\boldsymbol{X}_t = \boldsymbol{W}_t + a\boldsymbol{h}(t)$ denotes the Ornstein-Uhlenbeck process and "all $t \geq 0$" means that $\boldsymbol{\Theta}$ is smooth (and strictly positive) for $t \geq 0$. We should bear in mind that $\boldsymbol{\Theta}$ depends on a. The above comparison is based on the presence or absence of one multiplicative factor G_a. It is a "good" factor in that it restores smoothness which is otherwise broken at $t = 1/2a$; if G_a is neglected in (14), the solution becomes short-lived. On the other hand, if we insert $\widehat{G}_a$, which is valid as a martingale in $t < \sqrt{2}/a$, into (13), we obtain (22) below. (We are using the Maruyama-Girsanov theorem as a pull-back to retrieve the Navier-Stokes equations from the Leray equations.) The expression (22) does make sense as a path-integral equation on the same time interval.

Passing to the limit $a \to 0$ is equivalent to assume that there is no blowup to the Navier-Stokes equation because $t_* = 1/2a \to \infty$.§ In that limit we find formally

$$\lim_{a\to 0}\left(\boldsymbol{\Theta} - \mathbb{E}\big[\boldsymbol{F}[\boldsymbol{\Theta}](\boldsymbol{W}_t)\big]\right) = 0,$$

and

$$\lim_{a\to 0}\frac{1}{a}\left(\boldsymbol{\Theta} - \mathbb{E}\big[\boldsymbol{F}[\boldsymbol{\Theta}](\boldsymbol{W}_t)\big]\right) = \mathbb{E}\big[\langle D\boldsymbol{F}[\boldsymbol{\Theta}](\boldsymbol{W}_t), \boldsymbol{h}\rangle\big]. \tag{15}$$

The expression (15) follows from a definition of Malliavin derivative

$$\left.\frac{\partial}{\partial\epsilon}\right|_{\epsilon=0}; \ \ \mathbb{E}\big[\boldsymbol{F}[\boldsymbol{\Theta}](\boldsymbol{W}_t + \epsilon\boldsymbol{h})\big] = \mathbb{E}\big[\boldsymbol{F}[\boldsymbol{\Theta}](\boldsymbol{W}_t)G_\epsilon\big],$$

which resembles an elementary vectorial formula

$$\boldsymbol{h}\cdot\nabla f = \lim_{\epsilon\to 0}\frac{f(\boldsymbol{x}+\epsilon\boldsymbol{h}) - f(\boldsymbol{x})}{\epsilon} = \left.\frac{d}{d\epsilon}f(\boldsymbol{x}+\epsilon\boldsymbol{h})\right|_{\epsilon=0}.$$

We recall a formal definition of the Malliavin derivative $\boldsymbol{h}^*$ by $\frac{d}{d\epsilon}\boldsymbol{F}(\boldsymbol{W}_t + \epsilon\boldsymbol{h})\big|_{\epsilon=0} = \langle\boldsymbol{h}, \boldsymbol{h}^*\rangle$, where $\boldsymbol{h} \in \mathrm{H}$, $\boldsymbol{h}^* \in \mathrm{H}^*$ and $\langle,\rangle$ denotes an inner-product, H the Cameron-Martin space and H^* its dual space. The dual $\boldsymbol{h}^*$ is denoted by $D\boldsymbol{F}$.

§ It is known that no blowup can happen at very late times of order $T = O\left(E_0^2/\nu^5\right)$, where E_0 is the initial kinetic energy [13, 20]. Hence what $a \to 0$ actually means is $a \to O\left(\nu^5/E_0^2\right)$.

We next consider the case where a is not too small and characterise the difference in an additive fashion. We have

$$\underbrace{\boldsymbol{\Theta} - \mathbb{E}\big[\boldsymbol{F}[\boldsymbol{\Theta}](\boldsymbol{W}_t)\big]}_{\text{Navier-Stokes part}} = \mathbb{E}\big[\boldsymbol{F}[\boldsymbol{\Theta}](\boldsymbol{W}_t + a\boldsymbol{h}(t)) - \boldsymbol{F}[\boldsymbol{\Theta}](\boldsymbol{W}_t)\big] \tag{16}$$

$$= \mathbb{E}\big[\boldsymbol{F}[\boldsymbol{\Theta}](\boldsymbol{W}_t)(G_a - 1)\big], \tag{17}$$

$$\equiv \mathbb{E}\big[\langle D\boldsymbol{F}[\boldsymbol{\Theta}](\boldsymbol{W}_t + \mu\boldsymbol{h}(t)), a\boldsymbol{h}\rangle\big], \quad 0 < \mu < a, \tag{18}$$

which is valid for $t < \sqrt{2}/a$. Applying the usual mean-value theorem to G_a, we find

$$\frac{G_a - 1}{a} = \left.\frac{\partial G_a}{\partial a}\right|_{a=\mu}, \quad 0 < {}^{\exists}\mu < a,$$

where

$$\left.\frac{\partial G_a}{\partial a}\right|_{a=\mu} = \left(\int_0^t \boldsymbol{b}(\boldsymbol{W}_s)\cdot d\boldsymbol{W}_s - \mu\int_0^t |\boldsymbol{b}(\boldsymbol{W}_s)|^2 ds\right) G_\mu.$$

The equation (18) can be regarded as a result of an application of "the mean-value theorem"‖ to (16), whose precise meaning is given by (17). The equation (18) shows that the Leray equations have an extra additive term in the form of the Malliavin H-derivative, on top of the Navier-Stokes equations.

To summarise, we have for finite a

$$\frac{1}{a}\left(\boldsymbol{\Theta} - \mathbb{E}\big[\boldsymbol{F}[\boldsymbol{\Theta}](\boldsymbol{W}_t)\big]\right) = \mathbb{E}\big[\langle D\boldsymbol{F}[\boldsymbol{\Theta}](\boldsymbol{W}_t + \mu\boldsymbol{h}(t)), \boldsymbol{h}\rangle\big] \tag{19}$$

$$= \mathbb{E}\left[\boldsymbol{F}[\boldsymbol{\Theta}](\boldsymbol{W}_t)\left.\frac{\partial G_a}{\partial a}\right|_{a=\mu}\right]. \tag{20}$$

Note that we recover (15) in the limit $a \to 0$, because $\mu \to 0$ in that limit.

3.2. Navier-Stokes equations

Now we turn our attention to the Navier-Stokes equation of the form (11) and carry out an analysis in a parallel fashion. By assumption, it has a short-lived solution $\boldsymbol{\theta}$ for $t < 1/2a (= t_*)$, which satisfies

$$\boldsymbol{\theta} = \mathbb{E}\big[\boldsymbol{F}[\boldsymbol{\theta}](\boldsymbol{W}_t)\big], \quad 0 \le t < \frac{1}{2a} \tag{21}$$

$$= \mathbb{E}\left[\boldsymbol{F}[\boldsymbol{\theta}](\boldsymbol{W}_t + a\boldsymbol{h}(t))\widehat{G}_a\right], \tag{22}$$

where the Maruyama-Girsanov theorem has been applied with the density

$$\widehat{G}_a = \exp\left(-a\int_0^t \boldsymbol{b}(\boldsymbol{W}_s)\cdot d\boldsymbol{W}_s - \frac{a^2}{2}\int_0^t |\boldsymbol{b}(\boldsymbol{W}_s)|^2 ds\right).$$

‖ This is reminiscent of an application of the elementary mean-value theorem $f(x+a) = f(x) + af'(x+\mu)$, $0 < {}^{\exists}\mu < a$.

As above, we have

$$\underbrace{\boldsymbol{\theta} - \mathbb{E}\big[\boldsymbol{F}[\boldsymbol{\theta}](\boldsymbol{W}_t + a\boldsymbol{h}(t))\big]}_{\text{Leray part}} = \mathbb{E}\big[\boldsymbol{F}[\boldsymbol{\theta}](\boldsymbol{W}_t) - \boldsymbol{F}[\boldsymbol{\theta}](\boldsymbol{W}_t + a\boldsymbol{h}(t))\big] \tag{23}$$

$$= \mathbb{E}\Big[\boldsymbol{F}[\boldsymbol{\theta}](\boldsymbol{W}_t + a\boldsymbol{h}(t))(\widehat{G}_a - 1)\Big] \tag{24}$$

$$= -\mathbb{E}\big[\langle D\boldsymbol{F}[\boldsymbol{\theta}](\boldsymbol{W}_t + \mu'\boldsymbol{h}(t)), a\boldsymbol{h}\rangle\big], \tag{25}$$

where

$$\frac{\widehat{G}_a - 1}{a} = \left.\frac{\partial \widehat{G}_a}{\partial a}\right|_{a=\mu'}, \quad 0 < {}^{\exists}\mu' < a$$

$$\left.\frac{\partial \widehat{G}_a}{\partial a}\right|_{a=\mu'} = -\left(\int_0^t \boldsymbol{b}(\boldsymbol{W}_s)\cdot d\boldsymbol{W}_s + \mu' \int_0^t |\boldsymbol{b}(\boldsymbol{W}_s)|^2 ds\right)\widehat{G}_{\mu'}.$$

In short, we find

$$\frac{1}{a}\left(\boldsymbol{\theta} - \mathbb{E}\big[\boldsymbol{F}[\boldsymbol{\theta}](\boldsymbol{W}_t + a\boldsymbol{h}(t))\big]\right) = -\mathbb{E}\big[(D\boldsymbol{F}[\boldsymbol{\theta}](\boldsymbol{W}_t + \mu'\boldsymbol{h}(t)), \boldsymbol{h})\big] \tag{26}$$

$$= \mathbb{E}\left[\boldsymbol{F}[\boldsymbol{\theta}](\boldsymbol{W}_t)\left.\frac{\partial \widehat{G}_a}{\partial a}\right|_{a=\mu'}\right]. \tag{27}$$

In the limit $a \to 0$, we have formally

$$\lim_{a\to 0}\frac{1}{a}\left(\boldsymbol{\theta} - \mathbb{E}\big[\boldsymbol{F}[\boldsymbol{\theta}](\boldsymbol{W}_t + a\boldsymbol{h}(t))\big]\right) = -\mathbb{E}\big[(D\boldsymbol{F}[\boldsymbol{\theta}](\boldsymbol{W}_t), \boldsymbol{h})\big].$$

The equation (26) shows that the Navier-Stokes equations have an extra additive term in the form of the Malliavin H-derivative on top of the Leray equations. This time, it serves as a "bad" term because its presence makes otherwise long-lived solution short-lived.

Overall, we conclude that the presence or absence of the term $D\boldsymbol{F}$ would change the property of solutions drastically, if a solution to the Navier-Stokes equations breaks down in finite time.

4. Summary

We have compared the Navier-Stokes equations

$$\boldsymbol{\theta} = \mathbb{E}\big[\boldsymbol{F}[\boldsymbol{\theta}](\boldsymbol{W}_t)\big], \quad 0 \le t < \frac{1}{2a}.$$

with the dynamically-scaled Leray equations

$$\boldsymbol{\Theta} = \mathbb{E}\big[\boldsymbol{F}[\boldsymbol{\Theta}](\boldsymbol{W}_t + a\boldsymbol{h}(t))\big], \quad t \ge 0.$$

The Leray equations can be viewed as a distorted version of the Navier-Stokes equations

$$\boldsymbol{\Theta} = \mathbb{E}\big[\boldsymbol{F}[\boldsymbol{\Theta}](\boldsymbol{W}_t) + \langle D\boldsymbol{F}[\boldsymbol{\Theta}](\boldsymbol{W}_t + \mu\boldsymbol{h}(t)), a\boldsymbol{h}\rangle\big], \quad 0 \le t < \frac{\sqrt{2}}{a}.$$

If the H-derivative term $D\boldsymbol{F}$ is neglected, the equations would have a smooth solution only for $0 \le t < 1/2a \left(< \sqrt{2}/a\right)$. That is, the life-span must shrink at least by a factor of $2\sqrt{2}$.

Alternatively, the Navier-Stokes equations can be viewed as a distorted version of the Leray equations

$$\boldsymbol{\theta} = \mathbb{E}\big[\boldsymbol{F}[\boldsymbol{\theta}](\boldsymbol{W}_t + a\boldsymbol{h}(t)) - \langle D\boldsymbol{F}[\boldsymbol{\theta}](\boldsymbol{W}_t + \mu'\boldsymbol{h}(t)), a\boldsymbol{h}\rangle\big], \quad 0 \le t < \frac{1}{2a}.$$

If the H-derivative term is neglected, the equations would have a smooth solution for ${}^{\forall}t \ge 0$. Whether these are actually possible or not depend on the detailed properties of $\boldsymbol{F}$ beyond scaling.

The simplified exposition of quasi-invariance makes a clearer comparison between the Navier-Stokes and the Leray equations. A similar comparison in terms of the Malliavin derivatives can be done without introducing the Cole-Hopf transform. However, with the Cole-Hopf transform, i) we have only one G to take into account and ii) we can use the Feynman-Kac formula in its original form. Hence we believe that the current approach is the simplest way to characterise quasi-invariance.

In comparing the two equations before and after dynamic scaling, we recall that the closer they are, the more difficult it is for their solutions to behave in drastically different manners, i.e. one short-lived and the other long-lived. As the functional $\boldsymbol{F}$ is given in a fully explicit form, it seems worth investigating which of its specific properties, if any, can conclude to global regularity, e.g. by contraction. It is hoped that non-trivial use of stochastic analysis will improve understanding the Navier-Stokes equations.

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Appendix A. Feynman-Kac formula for time-dependent potential

For given $f_j(\boldsymbol{x}, t)$, a number of different representations are available for the (unique) solution to (10). To distinguish them properly, we assume here that Brownian motion starts from the origin $\boldsymbol{W}_0 = 0$, as opposed to the convention $\boldsymbol{W}_0 = \boldsymbol{x}$ adopted in the main text.

The expression (11)

$$\theta_j(\boldsymbol{x}, t) = \mathbb{E}\left[\theta_j(\boldsymbol{x} + \boldsymbol{W}_t, 0)\exp\left(\int_0^t f_j(\boldsymbol{x} + \boldsymbol{W}_s, s)ds\right)\right] \quad \text{(no summation)} \qquad \text{(A1)}$$

can be obtained by applying the time-dependent Trotter formula, see Section 11.2 of [27].

Another form

$$\theta_j(\boldsymbol{x}, t) = \mathbb{E}\left[\theta_j(\boldsymbol{x} + \boldsymbol{W}_t, 0)\exp\left(\int_0^t f_j(\boldsymbol{x} + \boldsymbol{W}_s, t - s)ds\right)\right] \quad \text{(no summation)} \qquad \text{(A2)}$$

may be found in [9].

Yet another form

$$\theta_j(\boldsymbol{x},t) = \mathbb{E}\left[\theta_j(\boldsymbol{x}+\boldsymbol{W}_t,0)\exp\left(\int_0^t f_j(\boldsymbol{x}+\boldsymbol{W}_t-\boldsymbol{W}_s,s)ds\right)\right] \quad \text{(no summation)} \tag{A3}$$

may be found in [10]. The expression (A3) can be extended to the case where the potential term f_j itself is stochastic [5]. If alternative forms are to be used, we should change all the arguments in $f_j(\cdot,\cdot)$ accordingly.

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