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Cherednik operators and Ruijsenaars-Schneider model at infinity

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Abstract. Heckman introduced N operators on the space of polynomials in N variables, such that these operators form a covariant set relative to permutations of the operators and variables, and such that Jack symmetric polynomials are eigenfunctions of the power sums of these operators. We introduce the analogues of these N operators for Macdonald symmetric polynomials, by using Cherednik operators. The latter operators pairwise commute, and Macdonald polynomials are eigenfunctions of their power sums. We compute the limits of our operators at $N \to \infty$. These limits yield a Lax operator for Macdonald symmetric functions.

Introduction

The Jack polynomials constitute a distinguished basis in the space of symmetric polynomials in N variables $x_1, \ldots, x_N$. They also depend on one parameter. These polynomials were first introduced in [5] but also appeared independently in [18] as eigenfunctions of a family of commuting differential operators playing the role of Hamiltonians of a quantum version of the classical Calogero-Moser model. The latter is a completely integrable system of one-dimensional particles interacting via a special two-particle potential [2, 9].

The Macdonald polynomials [8] are deformations of the Jack polynomials. They depend on two parameters usually denoted by q and t . These polynomials are eigenfunctions of certain commuting finite-difference operators, which are commuting Hamiltonians of the quantum Ruijsenaars-Schneider model [13, 14]. The latter model is a relativistic version of the quantum Calogero-Moser model.

There are two different constructions of commuting Hamiltonians. First, there is an explicit formula known as *Sekiguchi-Debiard determinant* in the Jack case, or *Macdonald determinant* in the Macdonald case [8]. The second construction is based on the theory of Hecke algebras. Namely, the commuting Hamiltonians in the Jack case can be obtained as power sums of the *Dunkl operators* representing the degenerate affine Hecke algebra. Respectively, the commuting Hamiltonians in the Macdonald case can be obtained as power sums of the *Cherednik operators* which represent the non-degenerate affine Hecke algebra [3].

On the other hand, studying the classical counterparts of the above mentioned quantum integrable systems is based on using *Lax matrices* whose characteristic determinants serve as generating functions of the commuting Hamiltonians. For the Jack polynomials, a quantum version $\mathcal{L}$ of the Lax matrix was used in [19] to produce an alternative formula for the Hamiltonians. Instead of a deteminantal generating function, they used a matrix element of the resolvent $(u-\mathcal{L})^{-1}$ taken relative to a particular vector and covector pair. Here u is a formal variable.

In the case of Jack polynomials, the similarity between the structures of the matrix $\mathcal{L}$ and of the Dunkl operators is rather striking. However, in the case of Macdonald polynomials the Lax matrix of the quantum Ruijsenaars-Schneider model has a structure different from that of the Cherednik operators. Moreover, a resolvent type formula for the Hamiltonians was unknown in the latter case. In the present article we solve this problem. We then use our solution to study the commuting Hamiltonians for the Macdonald polynomials when the number N of their variables tends to infinity. Thus we continue our recent works [10,11].

At $N \to \infty$ the Jack polynomials become symmetric functions in infinitely many variables $x_1, x_2, \ldots$. By using the Lax matrix formalism, in [10] we have constructed a family of pairwise commuting operators such that Jack symmetric functions are their eigenvectors. In [10] we expressed these commuting operators in terms of the power sums $x_1^n + x_2^n + \ldots$ where $n = 1, 2, \ldots$.

The Jack symmetric functions can be regarded as degenerations of Macdonald symmetric functions. In [11] we extended the results of [10] to the latter setting. In particular, by again using the Lax matrix formalism we constructed a family of pairwise commuting operators such that the Macdonald symmetric functions are their eigenvectors. In [11] we expressed these commuting operators in terms of the Hall-Littlewood symmetric functions of the variables $x_1, x_2, \ldots$ and of the parameter t. These expressions involve only the Hall-Littlewood symmetric functions corresponding to partitions with one part, see Subsection 1.2 below.

Soon after [10] was published, A. N. Sergeev and A. P. Veselov communicated to us their remarkable works [16,17] where in particular they found essentially the same commuting operators as we did in [10]. Their approach was different however. They first computed the limits at $N \to \infty$ of the Heckman operators [4] acting on all polynomials in the variables $x_1, \ldots, x_N$. These N operators do not commute in general. But the restrictions of the power sums of these N operators to the space of symmetric polynomials do commute. Moreover, Jack symmetric polynomials are eigenfunctions of these restrictions. Jack symmetric functions are then eigenfunctions of the limits of these restrictions at $N \to \infty$.

Here we extend this approach from Jack to Macdonald symmetric functions. It has been discovered by I. V. Cherednik [3] that the Macdonald polynomials in the variables $x_1, \ldots, x_N$ are eigenfunctions of power sums of some N commuting operators, acting on all polynomials in these variables. These operators are called the Cherednik operators, see our Subsection 2.2 for their definition. It has been also known [15] that the Cherednik operators have limits at $N \to \infty$. However, explicit expressions for these limits are unknown. We offer a solution to this open problem by firstly introducing for the Macdonald polynomials the analogues of non-commuting Heckman operators, see our Subsection 2.4. These analogues act on the rational functions of $x_1, \ldots, x_N$ and are denoted by $Z_1, \ldots, Z_N$. They are related to the Cherednik operators by Proposition 2.4. The principal property of the operators $Z_1, \ldots, Z_N$ is stated as Theorem 2.5, see also Proposition 2.2.

An explanation is needed here regarding our scheme of referring to lemmas, propositions, theorems and corollaries. When referring to these, we indicate the subsections where they respectively appear. There is no more than one of each of these in every subsection, so our scheme should cause no confusion. For example, Proposition 2.2 is the only proposition that appears in Subsection 2.2.

In Subsection 2.6 we reformulate Theorem 2.5 by introducing a certain $N\times N$ matrix $\mathcal{Z}$ with operator entries acting on the rational functions of $x_1,\ldots,x_N$. It is closely related to the Lax matrix of the classical Ruijsenaars-Schneider model [13]. To be precise, let γ_i be the operator defined by (2.1). In the classical limit $q\to 1$, when the canonical commutation relation $\gamma_i\, x_i = q^{-1} x_i\,\gamma_i$ degenerates to the Poisson bracket $\{\gamma_i\,,x_i\} = -\,\gamma_i\,x_i$, our $\mathcal{Z}$ degenerates to this Lax matrix up to a change of variables and up to conjugation by a diagonal matrix. In the same classical limit, the Macdonald determinant (2.3) degenerates to the characteristic determinant of the Lax matrix. Thus we have found a way to derive a quantum analogue of this Lax matrix directly from the Cherednik operators.

It is well known how the quantum Hamiltonians [14] of the trigonometric Ruijsenaars-Schneider model are related to the Macdonald operators (2.4), see for instance [7]. But our generating series (2.32) for quantum Hamiltonians differs from the Macdonald determinant and is new. A similar resolvent type expression was used for the quantum Calogero-Moser model in [19]. The eigenstates of the latter model are the Jack symmetric polynomials. The limit at $N\to\infty$ of the generalisation of that model to particles with spin has been studied in [6] as another extension of our work [10].

Following the approach of [16, 17] in Subsection 3.1 of our article we compute the limits at $N\to\infty$ of the operators $Z_1,\ldots,Z_N$. Then we also compute the limit of the restriction of the operator sum (2.20) appearing in Theorem 2.5 to the space of symmetric polynomials in $x_1,\ldots,x_N$. This limit is a formal power series in another variable u with operator coefficients acting on the symmetric functions of $x_1,x_2,\ldots$. After renormalisation and a change of the variable u , this limit becomes the same generating series of the pairwise commuting operators as we constructed in [11]. For details, see Subsections 3.2 and 3.3 here.

In this article we generally keep to the notation of the book [8] for symmetric functions. When using results from [8] we simply indicate their numbers within the book. For example, the statement (6.9) from Chapter I of the book will be referred to as [I.6.9] assuming it is from [8].

1. Symmetric functions

1.1. Standard symmetric functions. Fix any field $\mathbb{F}$. For every positive integer N denote by Λ_N the $\mathbb{F}$-algebra of symmetric polynomials in N variables $x_1,\ldots,x_N$. The algebra Λ_N is graded by the polynomial degree. The substitution $x_N=0$ defines a homomorphism $\Lambda_N\to\Lambda_{N-1}$ preserving the degree. Here $\Lambda_0=\mathbb{F}$. The inverse limit of the sequence

$$\Lambda_1\leftarrow\Lambda_2\leftarrow\ldots$$

in the category of graded algebras is denoted by Λ . Note that we get a canonical homomorhism $\Lambda\to\Lambda_N$. The elements of the algebra Λ are called *symmetric functions*. Following [8] we now will introduce some standard bases of Λ .

Let $\lambda=(\lambda_1,\lambda_2,\ldots)$ be any partition of $0,1,2,\ldots$. The number of non-zero parts is called the *length* of λ and is denoted by $\ell(\lambda)$. If $\ell(\lambda)\leqslant N$ then the sum of all distinct monomials obtained by permuting the N variables in $x_1^{\lambda_1}\dots x_N^{\lambda_N}$ is denoted by $m_\lambda(x_1,\dots,x_N)$. The symmetric polynomials $m_\lambda(x_1,\dots,x_N)$ with $\ell(\lambda)\leqslant N$ form a basis of the vector space Λ_N. By definition, for $\ell(\lambda)\leqslant N$

$$m_\lambda(x_1,\dots,x_N) \;=\; \sum_{1\leqslant i_1<\ldots<i_k\leqslant N}\;\sum_{\sigma\in\mathfrak{S}_k} c_\lambda^{-1}\; x_{i_{\sigma(1)}}^{\lambda_1}\dots x_{i_{\sigma(k)}}^{\lambda_k}$$

where we write k instead of $\ell(\lambda)$. Here $\mathfrak{S}_k$ is the symmetric group permuting the numbers $1,\dots,k$ and

$$c_\lambda = k_1!\,k_2!\,\dots$$

if $k_1,k_2,\dots$ are the respective multiplicites of the parts $1,2,\dots$ of λ. Further,

$$m_\lambda(x_1,\dots,x_{N-1},0) \;=\; \begin{cases} m_\lambda(x_1,\dots,x_{N-1}) & \text{if} \quad \ell(\lambda)<N\,; \\ 0 & \text{if} \quad \ell(\lambda)=N\,. \end{cases}$$

Hence for any fixed partition λ the sequence of polynomials $m_\lambda(x_1,\dots,x_N)$ with $N\geqslant\ell(\lambda)$ has a limit in Λ. This limit is called the *monomial symmetric function* corresponding to λ. Simply omitting the variables, we will denote the limit by m_λ. With λ ranging over all partitions of $0,1,2,\dots$ the symmetric functions m_λ form a basis of the vector space Λ. Note that if $\ell(\lambda)=0$ then we set $m_\lambda=1$.

We will be also using another standard basis of the vector space Λ. For each $n=1,2,\dots$ denote $p_n(x_1,\dots,x_N)=x_1^n+\ldots+x_N^n$. When the index n is fixed the sequence of symmetric polynomials $p_n(x_1,\dots,x_N)$ with $N=1,2,\dots$ has a limit in Λ, called the *power sum symmetric function* of degree n. We will denote it by p_n. More generally, for any partition λ put

$$p_\lambda = p_{\lambda_1}\dots p_{\lambda_k}$$

where $k=\ell(\lambda)$ as above. The elements p_λ form another basis of Λ. Equivalently, the elements $p_1,p_2,\dots$ are free generators of the commutative algebra Λ over $\mathbb{F}$.

In this article we will be using the *natural ordering* of partitions. By definition, here $\lambda\geqslant\mu$ if λ and μ are partitions of the same number and

$$\lambda_1\geqslant\mu_1,\;\;\lambda_1+\lambda_2\geqslant\mu_1+\mu_2,\;\;\dots\;\;.$$

This is a partial ordering. Note that by [I.6.9] any monomial symmetric function m_μ is a linear combination of the symmetric functions p_λ where $\lambda\geqslant\mu$.

Define a bilinear form $\langle\;,\;\rangle$ on Λ by setting for any two partitions λ and μ

$$\langle\, p_\lambda\,, p_\mu\rangle = k_\lambda\,\delta_{\lambda\mu} \quad \text{where} \quad k_\lambda = 1^{\,k_1}k_1!\;2^{\,k_2}k_2!\;\dots \tag{1.1}$$

in the above notation. This form is obviously symmetric and non-degenerate. We will indicate by the superscript $^\perp$ the operator conjugation relative to this form. In particular, by (1.1) for the operator conjugate to the multiplication in Λ by p_n with $n\geqslant1$ we have

$$p_n^\perp = n\,\partial/\partial\,p_n\,. \tag{1.2}$$

Next put

$$e_n(x_1,\dots,x_N) \;=\; \sum_{1\leqslant i_1<\ldots<i_n\leqslant N} x_{i_1}\dots x_{i_n}\,.$$

For any fixed n the sequence of the symmetric polynomials $e_n(x_1,\ldots,x_N)$ with $N=1,2,\ldots$ has a limit in Λ, denoted by e_n and called the *elementary symmetric function* of degree n. We will also use a formal power series in another variable v,

$$E(v)=1+e_1v+e_2v^2+\ldots=\prod_{i\geqslant1}(1+x_iv)\,. \tag{1.3}$$

By taking logarithms of the left and right hand side of the above display and then exponentiating,

$$E(v)\,=\,\exp\Big(-\sum_{n\geqslant1}\frac{p_n}{n}(-v)^n\Big)\,. \tag{1.4}$$

Also put

$$h_n(x_1,\ldots,x_N)\,=\sum_{\ell(\lambda)\leqslant N}m_\lambda(x_1,\ldots,x_N)$$

where the sum is taken over partitions λ of n. Then the sequence of symmetric polynomials $h_n(x_1,\ldots,x_N)$ with $N=1,2,\ldots$ has a limit in Λ, denoted by h_n and called the *complete symmetric function* of degree n. By [I.2.6] for the series

$$H(v)=1+h_1v+h_2v^2+\ldots\,.$$

we have the relation

$$E(-v)\,H(v)=1\,. \tag{1.5}$$

Hence (1.4) implies

$$H(v)\,=\,\exp\Big(\sum_{n\geqslant1}\frac{p_n}{n}v^n\Big)\,. \tag{1.6}$$

The elements $h_1,h_2,\ldots$ as well as the elements $e_1,e_2,\ldots$ are free generators of the commutative algebra Λ over the field $\mathbb{F}$. We will also use the *vertex operator*

$$H^{\perp}(v)=1+h_1^{\perp}v+h_2^{\perp}v^2+\ldots=\exp\Big(\sum_{n\geqslant1}\frac{p_n^{\perp}}{n}v^n\Big)\,. \tag{1.7}$$

It follows from (1.2) and (1.7) that for any $n=1,2,\ldots$ we have the equality

$$H^{\perp}(v)\,p_n=v^n+p_n\,. \tag{1.8}$$

It also follows from (1.2) and (1.7) that $H^{\perp}(v):\Lambda\to\Lambda[v]$ is a homomorphism of $\mathbb{F}$-algebras. See [I.5, Example 29] for both of the last two statements. Hence by applying $H^{\perp}(v)$ to any symmetric function in the variables $x_1,x_2,\ldots$ we get the same symmetric function but in the variables $v,x_1,x_2,\ldots$.

1.2. Hall-Littlewood symmetric functions. Let $\mathbb{F}$ be the field $\mathbb{Q}(t)$ with t a formal parameter. The Hall-Littlewood symmetric functions [III.2.11] are labelled by all partitions of $0,1,2,\ldots$ and constitute another remarkable basis of the vector space Λ over $\mathbb{F}$. In the present article we will use only the elements of this basis corresponding to the single part partitions $(1),(2),\ldots$. These elements will be denoted by $Q_1, Q_2, \ldots$ respectively. Their generating series is

$$Q(v) = E(-tv)\,H(v) = 1 + Q_1 v + Q_2 v^2 + \ldots\,.$$

By using (1.3) and (1.5) we get the relation

$$Q(v) \;=\; \prod_{i\geqslant 1} \frac{1 - t\,x_i\,v}{1 - x_i\,v} \tag{1.9}$$

while by using (1.4) and (1.6) we get the relation

$$Q(v) \;=\; \exp\Bigl(\,\sum_{n\geqslant 1} \frac{1-t^n}{n}\,p_n v^n\Bigr)\,. \tag{1.10}$$

1.3. Macdonald symmetric functions. Now let $\mathbb{F}$ be the field $\mathbb{Q}(q,t)$ where q and t are formal parameters. Then define a bilinear form $\langle\ ,\ \rangle_{q,t}$ on Λ by setting

$$\langle\, p_\lambda\,, p_\mu \rangle_{q,t} = k_\lambda\,\delta_{\lambda\mu} \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}} \tag{1.11}$$

for any partitions λ and μ. This form is symmetric and non-degenerate. If $q=t$, it specializes to the form defined by (1.1). We will indicate by the superscript * the operator conjugation relative to $\langle\ ,\ \rangle_{q,t}$. In particular, by (1.2) and (1.11)

$$p_n^* = \frac{1-q^n}{1-t^n}\,p_n^\perp$$

for any $n \geqslant 1$. Hence by using (1.10) we get

$$Q^*(v) = 1 + Q_1^* v + Q_2^* v^2 + \ldots \;=\; \exp\Bigl(\,\sum_{n\geqslant 1} \frac{1-q^n}{n}\,p_n^\perp v^n\Bigr)\,.$$

Note that by using (1.6), the latter identity can be rewritten as

$$Q^*(v) = H^\perp(v\,q)^{-1} H^\perp(v)\,. \tag{1.12}$$

Similarly to $H^\perp(v)$ the map $Q^*(v): \Lambda \to \Lambda[v]$ is a homomorphism of $\mathbb{F}$-algebras.

By [VI.4.7] there exists a unique family of elements $P_\lambda \in \Lambda$ such that

$$\langle\, P_\lambda\,, P_\mu \rangle_{q,t} = 0 \quad \text{for} \quad \lambda \neq \mu$$

and such that any P_λ equals m_λ plus a linear combination of the elements m_μ with $\mu < \lambda$ in the natural partial ordering. The elements $P_\lambda \in \Lambda$ are called the *Macdonald symmetric functions*.

By [VI.4.10] the canonical homomorphism $\Lambda \to \Lambda_N$ maps $P_\lambda \mapsto 0$ if $\ell(\lambda) > N$. If $\ell(\lambda) \leqslant N$ then the image of $P_\lambda \in \Lambda$ under the homomorphism $\Lambda \to \Lambda_N$ is the *Macdonald symmetric polynomial* usually denoted by $P_\lambda(x_1,\ldots,x_N)$. All these polynomials with $\ell(\lambda) = 0,1,\ldots,N$ form a basis of the vector space Λ_N over $\mathbb{F}$.

2. Cherednik operators

2.1. Macdonald operators. Let $\mathbb{F}=\mathbb{Q}(q,t)$ as in Subsection 1.3. For $i=1,\ldots,N$ the *inverse q-shift operator* γ_i acts on any rational function $f\in\mathbb{F}(x_1,\ldots,x_N)$ by

$$(\gamma_i f)(x_1,\ldots,x_N)=f(x_1,\ldots,q^{-1}x_i,\ldots,x_N)\,. \tag{2.1}$$

Denote by $\Delta(x_1,\ldots,x_N)$ the *Vandermonde polynomial* of N variables

$$\det\Big[x_i^{N-j}\Big]_{i,j=1}^{N}=\prod_{1\leqslant i<j\leqslant N}(x_i-x_j)\,.$$

Put

$$D_N(u)=\Delta(x_1,\ldots,x_N)^{-1}\cdot\det\Big[\,x_i^{N-j}\big(1+u\,t^{\,j-1}\gamma_i\big)\Big]_{i,j=1}^{N} \tag{2.2}$$

where u is another variable. The last determinant is defined as the alternated sum

$$\sum_{\sigma\in\mathfrak{S}_N}(-1)^{\sigma}\prod_{i=1}^{N}\big(\,x_i^{N-\sigma(i)}\big(1+u\,t^{\,\sigma(i)-1}\gamma_i\big)\big) \tag{2.3}$$

where as usual $(-1)^{\sigma}$ denotes the sign of permutation σ. In every product over $i=1,\ldots,N$ appearing in the alternated sum all the operator factors pairwise commute, hence their ordering does not matter. Note that $D_N(0)=1$.

By (2.2) the $D_N(u)$ is a polynomial of degree N in the variable u with operator coefficients. It also follows from (2.2) that these coefficients map the space Λ_N to itself. By [VI.3.3] for any $k=1,\ldots,N$ the coefficient of $D_N(u)$ at u^k equals

$$\sum_{|I|=k}S_I(x_1,\ldots,x_N)\prod_{i\in I}\gamma_i \tag{2.4}$$

where the sum is taken over all subsets I of $\{1,\ldots,N\}$ of size k, whereas

$$S_I(x_1,\ldots,x_N)=t^{\,k\,(k-1)/2}\prod_{\substack{i\in I\\ j\notin I}}\frac{x_i-t\,x_j}{x_i-x_j}\;.$$

Now for every $k=1,\ldots,N$ consider the restriction of the operator (2.4) to the space Λ_N. By [VI.4.16] all these restrictions to Λ_N pairwise commute. They are called the *Macdonald operators*. The Macdonald polynomials $P_\lambda(x_1,\ldots,x_N)$ with $\ell(\lambda)\leqslant N$ make a common eigenbasis of these operators. By [VI.4.15] the eigenvalue of $D_N(u)$ corresponding to any such eigenvector $P_\lambda(x_1,\ldots,x_N)$ is

$$\prod_{i=1}^{N}(1+u\,q^{-\lambda_i}t^{\,i-1})\,. \tag{2.5}$$

Note that our definition (2.2) of the $D_N(u)$ differs from [VI.3.2] by changing the parameters $q\mapsto q^{-1}$ and $t\mapsto t^{-1}$. However, by [VI.4.14] the Macdonald polynomials $P_\lambda(x_1,\ldots,x_N)$ are invariant under this change of their parameters. After this change, we also replaced the variable X used in the definition [VI.3.2] by $u\,t^{\,N-1}$. The reasons for these alterations will be explained in Subsection 3.1.

2.2. Cherednik operators. For $i, j = 1, \ldots, N$ with $i \neq j$ introduce the operator acting on the vector space $\mathbb{F}(x_1, \ldots, x_N)$

$$R_{ij} = 1 + \frac{(1-t)\,x_j}{x_i - x_j}\,(1 - \sigma_{ij}) = \frac{x_i - t\,x_j}{x_i - x_j} + \frac{(t-1)\,x_j}{x_i - x_j}\,\sigma_{ij} \tag{2.6}$$

where $\sigma_{ij} \in \mathfrak{S}_N$ acts by exchanging the variables x_i and x_j. It is immediately obvious from the definition (2.6) that the operator R_{ij} maps polynomials in the variables $x_1, \ldots, x_N$ to polynomials. Further, one can check that

$$t\,R_{ij}^{-1} = t + \frac{(t-1)\,x_j}{x_i - x_j}\,(1 - \sigma_{ij}) = \frac{t\,x_i - x_j}{x_i - x_j} + \frac{(1-t)\,x_j}{x_i - x_j}\,\sigma_{ij}\,.$$

The *Cherednik operators* $C_1, \ldots, C_N$ acting on $\mathbb{F}[x_1, \ldots, x_N]$ are then defined by

$$C_i = t^{\,i-1} R_{i,i+1} \ldots R_{iN}\,\gamma_i\,R_{1i}^{-1} \ldots R_{i-1,i}^{-1}\,. \tag{2.7}$$

These operators pairwise commute. In general, they do not map the space Λ_N to itself. However, any symmetric polynomial of the operators $C_1, \ldots, C_N$ with the coefficients from the field $\mathbb{F}$ does. Moreover by [3, Subsection 1.3.5] we have

Proposition. *The action of $D_N(u)$ on Λ_N coincides with that of the product*

$$\prod_{i=1}^{N} (1 + u\,C_i)\,. \tag{2.8}$$

In accord with the remark we made at the end of previous subsection, the operator C_i differs from the operator defined by [3, Equation 1.3.32] by changing the parameters $q \mapsto q^{-1}$ and $t \mapsto t^{-1}$. Our normalisation of C_i is also different.

2.3. Coherence property. We will use the following property of operators (2.7). For $k = 1, \ldots, N-1$ let

$$C_1^{(k)}, \ldots, C_{N-k}^{(k)} \tag{2.9}$$

be the Cherednik operators acting on $\mathbb{F}[x_{k+1}, \ldots, x_N]$ instead of $\mathbb{F}[x_1, \ldots, x_N]$.

Lemma. *The action of*

$$\prod_{i=k+1}^{N} (1 + u\,C_i) \tag{2.10}$$

on the space Λ_N coincides with the action of

$$\prod_{i=1}^{N-k} (1 + u\,t^{\,k}\,C_i^{(k)})\,. \tag{2.11}$$

Proof. First let us prove by the downward induction on $k = N, N-1, \ldots, 1, 0$ that the action of (2.10) on the space Λ_N coincides with the action of the product

$$\prod_{i=k+1}^{N} (1 + u\, t^{\,i-1} R_{i,i+1} \ldots R_{iN}\, \gamma_i\,) \tag{2.12}$$

where the factors corresponding to the indices $i = k+1, \ldots, N$ are arranged from left to right. If $k = N$ then neither of the products (2.10) and (2.12) has any factors, so the statement to prove is trivial. Now assume that our statement is already proved for some $k > 0$. Consider the product obtained from (2.10) by replacing the index k by $k-1$. By the induction assumption, the action on Λ_N of the so obtained product coincides with that of

$$(1 + u\, C_k) \prod_{i=k+1}^{N} (1 + u\, t^{\,i-1} R_{i,i+1} \ldots R_{iN}\, \gamma_i\,) \tag{2.13}$$

The last $k-1$ factors of the Cherednik operator C_k appearing in (2.13)

$$R_{1k}^{-1}, \ldots, R_{k-1,k}^{-1}$$

commute with $R_{i,i+1}$ and γ_i for any $i = k+1, \ldots, N$. They also act trivially on Λ_N. After removing these $k-1$ factors from C_k in (2.13) we get the product

$$(1 + u\, t^{\,k-1} R_{k,k+1} \ldots R_{kN}\, \gamma_k\,) \prod_{i=k+1}^{N} (1 + u\, t^{\,i-1} R_{i,i+1} \ldots R_{iN}\, \gamma_i\,)\,.$$

Thus we are making the induction step, and our statement is proved for any k.

By using this statement in the particular case when $k = 0$, the action of (2.8) on the space Λ_N coincides with the action of the product

$$\prod_{i=1}^{N} (1 + u\, t^{\,i-1} R_{i,i+1} \ldots R_{iN}\, \gamma_i\,)\,.$$

By applying the latter result to the set of operators (2.9) instead of $C_1, \ldots, C_N$ we obtain that for $0 < k < N$ the action of (2.11) on Λ_N coincides with that of

$$\prod_{i=1}^{N-k} (1 + u\, t^{\,i+k-1} R_{i+k,i+k+1} \ldots R_{i+k,N}\, \gamma_{i+k}\,)\,.$$

The last displayed product equals (2.12) by renaming $i+k$ to i. But we had also proved that the action of (2.10) on Λ_N coincides with the action of (2.12). $\square$

2.4. Covariant operators. For $i,j=1,\ldots,N$ with $i\neq j$ denote

$$A_{ij}=\frac{x_i-t\,x_j}{x_i-x_j}\quad\text{and}\quad B_{ij}=\frac{(\,t-1\,)\,x_j}{x_i-x_j}\tag{2.14}$$

so that by (2.6)

$$R_{ij}=A_{ij}+B_{ij}\,\sigma_{ij}\,.$$

Define operators $Z_1,\ldots,Z_N$ acting on $\mathbb{F}(x_1,\ldots,x_N)$ by setting $Z_i=W_i\gamma_i$ where

$$W_i\;=\prod_{l\neq i}A_{il}\;+\sum_{j\neq i}B_{ij}\Bigl(\prod_{l\neq i,j}A_{jl}\Bigr)\,\sigma_{ij}\,.\tag{2.15}$$

In general, these operators do not map polynomials in the variables $x_1,\ldots,x_N$ to polynomials. But by definition, these operators make a *covariant set* relative to the action of the symmetric group $\mathfrak{S}_N$ by permutations of the variables:

$$\sigma^{-1}Z_i\,\sigma=Z_{\sigma(i)}\quad\text{for}\quad\sigma\in\mathfrak{S}_N\,.\tag{2.16}$$

Note that for $N>1$ the operators $C_1,\ldots,C_N$ on $\mathbb{F}[x_1,\ldots,x_N]$ do not enjoy the covariance property. On the other hand, our $Z_1,\ldots,Z_N$ do not commute.

For $k=1,\ldots,N-1$ let $\Lambda_N^{(k)}\subset\mathbb{F}[x_1,\ldots,x_N]$ be the subspace of polynomials symmetric in the variables $x_{k+1},\ldots,x_N$. Then

$$\Lambda_N\subset\Lambda_N^{(1)}\subset\ldots\subset\Lambda_N^{(N-1)}=\mathbb{F}[x_1,\ldots,x_N]\,.$$

Now consider the Cherednik operator C_1 acting on $\mathbb{F}[x_1,\ldots,x_N]$. Our definition of the operator Z_1 originates from the following proposition.

Proposition. *The actions of the operators C_1 and Z_1 on $\Lambda_N^{(1)}$ coincide.*

Proof. We will prove that the action of C_1 on $\Lambda_N^{(k)}$ coincides with the action of

$$R_{12}\ldots R_{1k}\,\Bigl(\prod_{k<l\leqslant N}A_{1l}\;+\sum_{k<j\leqslant N}B_{1j}\Bigl(\prod_{\substack{k<l\leqslant N\\ l\neq j}}A_{jl}\Bigr)\,\sigma_{1j}\Bigr)\,\gamma_1\,.\tag{2.17}$$

We will use the downward induction on $k=N-1\,,\ldots,1$. Our proposition will be then obtained when $k=1$. If $k=N-1$ then by the definition (2.7) we have

$$C_1=R_{12}\ldots R_{1N}\,\gamma_1=R_{12}\ldots R_{1,N-1}\,(\,A_{1N}+B_{1N}\,\sigma_{1N}\,)\,\gamma_1$$

as required. Now assume that our statement is proved for some $k>1$. Since

$$\Lambda_N^{(k-1)}\subset\Lambda_N^{(k)}\tag{2.18}$$

we then know in particular that the action of C_1 on the space $\Lambda_N^{(k-1)}$ coincides with the action of the product (2.17). The latter product can be rewritten as

$$R_{12}\ldots R_{1,k-1}\;\times$$
$$(\,A_{1k}+B_{1k}\,\sigma_{1k}\,)\,\Bigl(\prod_{k<l\leqslant N}A_{1l}\;+\sum_{k<j\leqslant N}B_{1j}\Bigl(\prod_{\substack{k<l\leqslant N\\ l\neq j}}A_{jl}\Bigr)\,\sigma_{1j}\Bigr)\,\gamma_1\,.$$

In its turn, the expression in the last displayed line can be rewritten as

$$\Bigl(\,\prod_{k\leqslant l\leqslant N} A_{1l}\,+\,B_{1k}\Bigl(\,\prod_{k<l\leqslant N} A_{kl}\Bigr)\,\sigma_{1k}\,+$$
$$\sum_{k<j\leqslant N}(A_{1k}\,B_{1j}+B_{1k}\,B_{kj}\,\sigma_{1k})\,\Bigl(\,\prod_{\substack{k<l\leqslant N\\ l\neq j}} A_{jl}\Bigr)\,\sigma_{1j}\Bigr)\,\gamma_1\,.$$

Here none of the indices of the factor A_{jl} can be equal to 1 or k, because $j>k$ and $l>k$. Further, here $\sigma_{1k}\,\sigma_{1j}=\sigma_{1j}\,\sigma_{jk}$ where the factor σ_{jk} commutes with the operator γ_1 on $\mathbb{F}[x_1,\dots,x_N]$ and acts trivially on the subspace (2.18). Thus by the identity

$$A_{1k}\,B_{1j}+B_{1k}\,B_{kj}=B_{1j}\,A_{jk}$$

the action of the operator C_1 on $\Lambda_N^{(k-1)}$ coincides with the action of

$$R_{12}\dots R_{1,k-1}\,\times$$
$$\Bigl(\,\prod_{k\leqslant l\leqslant N} A_{1l}\,+\,B_{1k}\Bigl(\,\prod_{k<l\leqslant N} A_{kl}\Bigr)\,\sigma_{1k}\,+\sum_{k<j\leqslant N} B_{1j}\Bigl(\,\prod_{\substack{k\leqslant l\leqslant N\\ l\neq j}} A_{jl}\Bigr)\,\sigma_{1j}\Bigr)\,\gamma_1$$
$$=R_{12}\dots R_{1,k-1}\,\Bigl(\,\prod_{k-1<l\leqslant N} A_{1l}\,+\sum_{k-1<j\leqslant N} B_{1j}\Bigl(\,\prod_{\substack{k-1<l\leqslant N\\ l\neq j}} A_{jl}\Bigr)\,\sigma_{1j}\Bigr)\,\gamma_1\,.$$

Thus we have made the downward induction step. □

We have already noted that for any $i=1,\dots,N$ the Cherednik operator C_i maps the polynomials in the variables $x_1,\dots,x_N$ to polynomials. On the other hand, the operator Z_i commutes with those permutations of the variables that preserve x_i. By using these two observations when $i=1$, our proposition implies

Corollary. *Both operators C_1 and Z_1 map the space $\Lambda_N^{(1)}$ to itself.*

2.5. Main identity. Our main result of the current section is the theorem below. Define operators $U_1,\dots,U_N$ acting on $\mathbb{F}(x_1,\dots,x_N)$ by setting

$$U_i\,=\,(\,t-1)\,\Bigl(\,\prod_{l\neq i} A_{il}\Bigr)\,\gamma_i\,. \tag{2.19}$$

Similarly to $Z_1,\dots,Z_N$ the operators $U_1,\dots,U_N$ make a covariant set relative to the action of the group $\mathfrak{S}_N$ by permutations of the variables $x_1,\dots,x_N$:

$$\sigma^{-1}U_i\,\sigma=U_{\sigma(i)}\quad\text{for}\quad\sigma\in\mathfrak{S}_N\,.$$

Theorem. *The action of the ratio $D_N(u\,t)/D_N(u)$ on Λ_N coincides with the action of the sum*

$$1\,+\,u\sum_{i=1}^{N} U_i\,(1+u\,Z_i)^{-1}\,. \tag{2.20}$$

Proof. We will relate operators on the space $\mathbb{F}(x_1,\dots,x_N)$ by the symbol $\sim$ if their actions on the subspace Λ_N coincide. In Subsection 2.1 we already noted that the coefficients of the polynomial $D_N(u)$ map the space Λ_N to itself. Let us multiply by $D_N(u)$ on the right both the ratio and the sum appearing in our theorem, and then subtract $D_N(u)$ from the results. We get to prove the relation

$$D_N(u\,t)-D_N(u)\,\sim\,u\sum_{i=1}^{N}U_i\,(1+u\,Z_i)^{-1}\,D_N(u)\,. \tag{2.21}$$

In the notation of Subsection 2.1 the left hand side of the relation (2.21) equals

$$u\sum_{k=1}^{N}u^{k-1}(\,t^{\,k}-1)\sum_{|I|=k}S_I(x_1,\dots,x_N)\,\prod_{i\in I}\gamma_i\,.$$

Now consider the summand at the right hand side of (2.21) with the index $i=1$. By Proposition 2.2 the action of this summand on Λ_N coincides with that of

$$U_1\,(1+u\,Z_1)^{-1}\prod_{j=1}^{N}\,(1+u\,C_j)\,\sim\,U_1\prod_{j=2}^{N}\,(1+u\,C_j)\,\sim\,U_1\prod_{j=1}^{N-1}(1+u\,t\,C_j^{(1)})$$

where we used Proposition 2.4 and then Lemma 2.3 in the particular case $k=1$. Hence by applying Proposition 2.2 once again, but to the Cherednik operators

$$C_1^{(1)},\dots,C_{N-1}^{(1)}$$

instead of $C_1,\dots,C_N$ we obtain that the summand at the right hand side of the relation (2.21) with the index $i=1$ acts on Λ_N as

$$U_1\sum_{k=1}^{N}(u\,t)^{k-1}\sum_{|J|=k-1}S_J(x_2,\dots,x_N)\,\prod_{j\in J}\gamma_j\;=$$

$$\sum_{k=1}^{N}(u\,t)^{k-1}(\,t-1)\sum_{|J|=k-1}\Bigl(\,\prod_{l\neq 1}A_{1l}\Bigr)\,S_J(x_2,\dots,x_N)\,\gamma_1\prod_{j\in J}\gamma_j\,. \tag{2.22}$$

Here J ranges over all subsets of $\{2,\dots,N\}$ of size $k-1$. It follows that the summands at the right hand side of (2.21) with $i=2,\dots,N$ act on Λ_N as the operators obtained from (2.22) via conjugation by $\sigma_{12}\,,\dots,\sigma_{1N}$ respectively.

Thus the right hand side of (2.21) acts on Λ_N as the operator sum of the form

$$\sum_{k=1}^{N}u^{k-1}\sum_{|I|=k}T_I\,(x_1,\dots,x_N)\,\prod_{i\in I}\gamma_i$$

where I ranges over all subsets of $\{1,\dots,N\}$ of size k, and each $T_I\,(x_1,\dots,x_N)$ is a certain rational function of the variables $x_1,\dots,x_N$ over the field $\mathbb{Q}(t)$. To prove the relation (2.21) it now suffices to demonstrate that for each I

$$(\,t^{\,k}-1)\,S_I(x_1,\dots,x_N)=T_I\,(x_1,\dots,x_N)\,. \tag{2.23}$$

Moreover, because both sides of (2.21) are invariant under conjugation by the elements of $\mathfrak{S}_N$, it suffices to verify (2.23) only in the case when $I=\{1,\ldots,k\}$. Note that in the latter case the left hand side of (2.23) equals

$$(t^k-1)\,t^{k(k-1)/2}\prod_{\substack{1\leqslant j\leqslant k\\ k<l\leqslant N}}A_{jl}\,. \tag{2.24}$$

Now consider the right hand side of (2.23) in the case when $I=\{1,\ldots,k\}$. Let us denote it by T for short. The contribution to T from (2.22) corresponds to the set $J=\{2,\ldots,k\}$ and hence equals

$$t^{k-1}(t-1)\Big(\prod_{l\neq 1}A_{1l}\Big)\,t^{(k-1)(k-2)/2}\prod_{\substack{2\leqslant j\leqslant k\\ k<l\leqslant N}}A_{jl}\;=$$

$$(t-1)\Big(\prod_{1<l\leqslant k}A_{1l}\Big)\,t^{k(k-1)/2}\prod_{\substack{1\leqslant j\leqslant k\\ k<l\leqslant N}}A_{jl}\,. \tag{2.25}$$

If we conjugate (2.22) by any σ_{1i} with $i>1$, the result will make contribution to B only when $i\leqslant k$, and this contribution will correspond to $J=\{2,\ldots,k\}$. Indeed, then we will need J in (2.22) such that $\sigma_{1i}(\{1\}\sqcup J)=\{1,\ldots,k\}$. Hence for each index $i=2,\ldots,k$ we get a contribution to T

$$\sigma_{1i}\Big((t-1)\Big(\prod_{1<l\leqslant k}A_{1l}\Big)\,t^{k(k-1)/2}\prod_{\substack{1\leqslant j\leqslant k\\ k<l\leqslant N}}A_{jl}\Big)=$$

$$(t-1)\Big(\prod_{\substack{1\leqslant l\leqslant k\\ l\neq i}}A_{il}\Big)\,t^{k(k-1)/2}\prod_{\substack{1\leqslant j\leqslant k\\ k<l\leqslant N}}A_{jl}\,. \tag{2.26}$$

By dividing (2.24),(2.25),(2.26) by $(t-1)\,t^{k(k-1)/2}$ and by cancelling there all the common factors A_{jl} the relation (2.23) now reduces to the identity

$$\frac{t^k-1}{t-1}=\sum_{i=1}^{k}\prod_{\substack{1\leqslant j\leqslant k\\ j\neq i}}\frac{x_i-t\,x_j}{x_i-x_j}\;.$$

The latter identity is easy to verify, and we omit the details of verification. □

Consider the operator sum over $i=1,\ldots,N$ appearing in (2.20). Denote by $I_N(u)$ the restriction of this operator sum to the subspace $\Lambda_N\subset\mathbb{F}(x_1,\ldots,x_N)$. The $I_N(u)$ expands as a formal power series in u with coefficients acting on Λ_N. Our theorem means that the action of the coefficients of the series $1+u\,I_N(u)$ on Λ_N coincides with the action of the respective coefficients of $D_N(u\,t)/D_N(u)$. The latter ratio should be also expanded as a formal power series in u here.

The coefficients of the series $I_N(u)$ will be called the *quantum Hamiltonians* corresponding to the basis of Macdonald polynomials in the vector space Λ_N. In the next subsection we give another expression for $I_N(u)$ by using the resolvent of a certain $N\times N$ matrix with operator entries which act on $\mathbb{F}(x_1,\ldots,x_N)$.

2.6. Matrix resolvent. Take any $f \in \Lambda_N^{(1)}$ and consider the column vector

$$\mathcal{F} \;=\; \begin{bmatrix} f \\ \sigma_{12}(f) \\ \vdots \\ \sigma_{1N}(f) \end{bmatrix}$$

Now define a $N \times N$ matrix $\mathcal{Z}$ with operator entries acting on the vector space $\mathbb{F}(x_1, \ldots, x_N)$ as follows. The $i\,,j$-entry Z_{ij} of the matrix $\mathcal{Z}$ is defined by setting

$$Z_{ii} = \Big(\prod_{l \neq i} A_{il} \Big) \gamma_i \,, \tag{2.27}$$

$$Z_{ij} = B_{ij} \Big(\prod_{l \neq i,j} A_{jl} \Big) \gamma_j \quad \text{for} \quad i \neq j\,. \tag{2.28}$$

Then by using the definition (2.15) we have

$$Z_i \;=\; W_i \gamma_i \;=\; Z_{ii} \;+\; \sum_{j \neq i} Z_{ij}\, \sigma_{ij} \tag{2.29}$$

where for $j \neq i$ we also use the relation $\sigma_{ij}\, \gamma_i = \gamma_j\, \sigma_{ij}$. It follows from (2.29) that

$$\begin{bmatrix} Z_1(f) \\ Z_2\, \sigma_{12}(f) \\ \vdots \\ Z_N\, \sigma_{1N}(f) \end{bmatrix} \;=\; \mathcal{Z}\, \mathcal{F}\,. \tag{2.30}$$

Indeed, in its first entry the vector equality (2.30) holds by (2.29) with $i = 1$. If $i \neq 1$ then by using (2.29) we get

$$Z_i\, \sigma_{1i}(f) \;=\; Z_{ii}\, \sigma_{1i}(f) + Z_{i1}\, \sigma_{i1}\, \sigma_{1i}(f) + \sum_{j \neq 1,i} Z_{ij}\, \sigma_{ij}\, \sigma_{1i}(f) \;=$$
$$Z_{ii}\, \sigma_{1i}(f) + Z_{i1}(f) + \sum_{j \neq 1,i} Z_{ij}\, \sigma_{1j}(f) \;=\; Z_{i1}(f) + \sum_{j \neq 1} Z_{ij}\, \sigma_{1j}(f)$$

as required. Here for three pairwise distinct indices $1, i, j$ we also use the relations

$$\sigma_{ij}\, \sigma_{1i}(f) = \sigma_{1j}\, \sigma_{ij}(f) = \sigma_{1j}(f)\,.$$

By the the covariance property (2.16) of the operators $Z_1, \ldots, Z_N$ the column vector at the left hand side of (2.30) has a form similar to $\mathcal{F}$. Namely, it can be obtained by replacing the polynomial f in $\mathcal{F}$ by $Z_1(f)$. Here we use Corollary 2.4. By expanding $(1 + u\, Z_i)^{-1}$ for every $i = 1, \ldots, N$ as a formal power series in u and by repeatedly using the above arguments, we get the equality

$$\begin{bmatrix} (1 + u\, Z_1)^{-1}(f) \\ (1 + u\, Z_2)^{-1}\, \sigma_{12}(f) \\ \vdots \\ (1 + u\, Z_N)^{-1}\, \sigma_{1N}(f) \end{bmatrix} \;=\; (1 + u\, \mathcal{Z})^{-1}\, \mathcal{F}\,. \tag{2.31}$$

Now suppose $f \in \Lambda_N$ so that the polynomial f is symmetric in all the variables $x_1, \ldots, x_N$. Then

$$f = \sigma_{12}(f) = \ldots = \sigma_{1N}(f)$$

so that $\mathcal{F} = \mathcal{E} f$ where $\mathcal{E}$ is the column vector of size N with every entry being 1. Let $\mathcal{U}$ be the row vector of size N where the i-entry is the U_i defined by (2.19). By using (2.31) and the definition of the series $I_N(u)$ as given in Subsection 2.5

$$I_N(u)\, f = \mathcal{U}\,(\,1 + u\,\mathcal{Z}\,)^{-1}\,\mathcal{E}\, f\,.$$

Thus we have proved that the action of $I_N(u)$ on Λ_N coincides with the action of

$$\mathcal{U}\,(\,1 + u\,\mathcal{Z}\,)^{-1}\,\mathcal{E}\,. \tag{2.32}$$

Hence we now obtain the following corollary to Theorem 2.5.

Corollary. *The action of the ratio $D_N(u\,t)/D_N(u)$ on Λ_N coincides with that of*

$$1 \,+\, u\,\mathcal{U}\,(\,1 + u\,\mathcal{Z}\,)^{-1}\,\mathcal{E}\,.$$

3. Inverse limits

3.1. Limits of covariant operators. Let $\mathbb{F} = \mathbb{Q}(q,t)$ as before. We will find first the inverse limit at $N \to \infty$ of the restriction of the operator Z_1 to the subspace

$$\Lambda_N^{(1)} \subset \mathbb{F}(x_1, \ldots, x_N)\,. \tag{3.1}$$

By Proposition 2.4 the operator C_1 has the same restriction to $\Lambda_N^{(1)}$. The limit will be an operator acting on the space $\Lambda[v]$ and denoted simply by Z. To define the limit extend the canonical homomorphism $\Lambda \to \Lambda_N$ to a homomorphism

$$\pi_N \,:\, \Lambda[v] \,\to\, \Lambda_N^{(1)} \,:\, v \,\mapsto\, x_1\,.$$

Here

$$\pi_N \,:\, p_n \,\mapsto\, p_n(x_1, \ldots, x_N) \quad \text{for} \quad n = 1, 2, \,\ldots\,.$$

We will now define an operator Z on the vector space $\Lambda[v]$ explicitly. Denote by ξ and η the automorphisms of the $\mathbb{F}$-algebra $\Lambda[v]$ which act trivially on the subalgebra Λ but map the variable v to $q^{-1}v$ and $t\,v$ respectively. Thus ξ is the inverse q-shift of v while η is the usual t-shift. Next equip the vector space $\mathbb{F}[v]$ with the standard inner product so that $1, v, v^2, \ldots$ form an orthonormal basis. Denote by v° the operator on $\mathbb{F}[v]$ conjugate to multiplication by v. Explicitly,

$$v^\circ \,:\, v^n \,\mapsto\, \begin{cases} v^{n-1} & \text{if} \quad n > 0\,, \\ \;\;0 & \text{if} \quad n = 0\,. \end{cases}$$

Extend the operator v° from $\mathbb{F}[v]$ to $\Lambda[v]$ by Λ-linearity. The extension will still be denoted by v°. For every $f \in \Lambda$ extend from Λ to $\Lambda[v]$ by $\mathbb{F}[v]$-linearity the operator of multiplication by f and its conjugate operator $f^{\perp}$. Recall that the superscript $^{\perp}$ here indicates conjugation relative to the inner product (1.1). The conjugate f^* relative to the inner product (1.11) extends from Λ to $\Lambda[v]$ in the same way as $f^{\perp}$ does. Using these conventions, put $Z = W\gamma$ where

$$\gamma = \xi\, Q^*(v) \quad \text{and} \quad W = \eta\, Q(v^\circ)\,.$$

Theorem. *We have a commutative diagram of $\mathbb{F}$-linear mappings*

$$
\begin{array}{ccc}
\Lambda[v] & \xrightarrow{\;Z\;} & \Lambda[v] \\
\Big\downarrow{\scriptstyle \pi_N} & & \Big\downarrow{\scriptstyle \pi_N} \\
\Lambda_N^{(1)} & \xrightarrow[Z_1]{} & \Lambda_N^{(1)}
\end{array}
\tag{3.2}
$$

Proof. We will verify commutativity of the two diagrams obtained from (3.2) by replacing $Z\,,Z_1$ respectively by γ,γ_1 and $W\,,W_1$. Our theorem will then follow.

Firstly observe that the extended operator $Q^*(v):\Lambda[v]\to\Lambda[v]$ appearing in the definition of γ is a homomorphism of $\mathbb{F}$-algebras, and so is $\xi:\Lambda[v]\to\Lambda[v]$. Hence it suffices to show that the compositions $\pi_N\,\gamma$ and $\gamma_1\,\pi_N$ coincide on v and on p_n for $n=1,2,\,\ldots$. By applying the compositions to v we get the same result:

$$
v\;\underset{\gamma}{\longmapsto}\;q^{-1}v\;\underset{\pi_N}{\longmapsto}\;q^{-1}x_1\qquad\text{and}\qquad v\;\underset{\pi_N}{\longmapsto}\;x_1\;\underset{\gamma_1}{\longmapsto}\;q^{-1}x_1\,.
$$

To check the coincidence on any generator p_n note that by the identity (1.12)

$$
\gamma=\xi\,H^{\perp}(v\,q)^{-1}\,H^{\perp}(v)\;=H^{\perp}(v)^{-1}\,\xi\,H^{\perp}(v)\,.
$$

Hence due to (1.8) by applying $\pi_N\,\gamma$ and $\gamma_1\,\pi_N$ to p_n we also get the same result:

$$
p_n\;\underset{\gamma}{\longmapsto}\;q^{-n}v^n-v^n+p_n\;\underset{\pi_N}{\longmapsto}\;q^{-n}x_1+x_2^n+\ldots+x_N^n
$$

and

$$
p_n\;\underset{\pi_N}{\longmapsto}\;x_1^n+\ldots+x_N^n\;\underset{\gamma_1}{\longmapsto}\;q^{-n}x_1+x_2^n+\ldots+x_N^n\,.
$$

Now consider the compositions $\pi_N W$ and $W_1\pi_N$. By definition, the extended operator $W:\Lambda[v]\to\Lambda[v]$ commutes with multiplication by any $f\in\Lambda$. But

$$
W_1\;=\prod_{1<l\leqslant N}A_{1l}\;+\sum_{1<j\leqslant N}B_{1j}\,\Big(\prod_{\substack{1<l\leqslant N\\ l\neq j}}A_{jl}\Big)\,\sigma_{1j}
\tag{3.3}
$$

by (2.15). In particular, the restriction of the operator W_1 to the subspace (3.1) commutes with multiplication by $\pi_N(f)\in\Lambda_N$. Hence it suffices to show that the compositions $\pi_N W$ and $W_1\pi_N$ coincide on the elements $1,v,v^2,\,\ldots\;\in\Lambda[v]$. Let us use the generating series of these elements

$$
1+u\,v+u^2v^2+\ldots\;=\;\frac{1}{1-u\,v}
\tag{3.4}
$$

in the other variable u . By applying $\pi_N W$ to the series (3.4) we get

$$
\frac{1}{1-u\,v}\;\underset{W}{\longmapsto}\;\frac{Q(u)}{1-u\,t\,v}\;\underset{\pi_N}{\longmapsto}\;\frac{1}{1-u\,t\,x_1}\,\prod_{i=1}^{N}\frac{1-u\,t\,x_i}{1-u\,x_i}\;\;.
\tag{3.5}
$$

Here we employed the general fact that for any formal power series $G(u)$ with the coefficients from $\mathbb{F}$

$$G(v^\circ)\,\frac{1}{1-u\,v}=\frac{G(u)}{1-u\,v}\;.$$

We have also employed the relation (1.9) with the variable v replaced by u.

On the other hand, by applying $W_1\pi_N$ to the series (3.4) we get

$$\frac{1}{1-u\,v}\;\underset{\pi_N}{\longmapsto}\;\frac{1}{1-u\,x_1}\;\underset{W_1}{\longmapsto}\;\frac{1}{1-u\,x_1}\prod_{1<l\leqslant N}\frac{x_1-t\,x_l}{x_1-x_l}\;+$$

$$\sum_{1<j\leqslant N}\frac{(t-1)\,x_j}{(1-u\,x_j)\,(x_1-x_j)}\prod_{\substack{1<l\leqslant N\\ l\neq j}}\frac{x_j-t\,x_l}{x_j-x_l}\;.$$

Here we also used (2.14) and (3.3). It easy to verify that the results obtained in (3.5) and in the last two displayed lines are the same. Consider them as rational functions of u and assume that $x_1,\ldots,x_N\neq 0$. Then both rational functions vanish at $u=\infty$ and have poles only at $u=x_1^{-1},\ldots,x_N^{-1}$. All these poles are simple, and the corresponding residues of the two functions coincide. $\square$

Note that for any index $i=2,\ldots,N$ one can also consider the restriction of the operator Z_i to the subspace of $\mathbb{F}(x_1,\ldots,x_N)$ consisting of the polynomials in $x_1,\ldots,x_N$ symmetric in all the variables but x_i. By the covariance property (2.16) our Corollary 2.4 implies that the operator Z_i preserves this subspace. We could have defined the extension π_N of the homomorphism $\Lambda\to\Lambda_N$ from Λ to $\Lambda[v]$ by mapping the variable v to x_i instead of x_1. The image of π_N would be then the latter subspace of $\mathbb{F}(x_1,\ldots,x_N)$. The inverse limit of the restriction of Z_i to that subspace would be then the same operator Z acting on $\Lambda[v]$. This coincidence follows immediately from the property (2.16).

It is the change of parameters $q\mapsto q^{-1}$ and $t\mapsto t^{-1}$ in the original definition [VI.3.2] that allowed us to state the last theorem in terms of the Hall-Littlewood symmetric functions $Q_1,Q_2,\ldots$. Otherwise we would have to change $t\mapsto t^{-1}$ in the definition of the latter symmetric functions. The change of the variable X in [VI.3.2] and the corresponding choice of normalization of the operator C_i as in (2.7) and as in Proposition 2.2 ensure that every Z_i has a limit at $N\to\infty$.

3.2. Limits of quantum Hamiltonians. In this subsection we will find the inverse limits at $N\to\infty$ of the quantum Hamiltonians corresponding to the basis of Macdonald polynomials in Λ_N. These quantum Hamiltonians are defined as the operator coefficients of the series $I_N(u)$ acting on the vector space Λ_N, see the end of Subsection 2.5. We will denote by $I(u)$ the inverse limit of the series $I_N(u)$.

The coefficients of the series $I(u)$ will be certain operators $\Lambda\to\Lambda[w]$ where w is yet another formal variable. We will then eliminate the dependence of the coefficients on w by renormalising the series $I(u)$. Hence the coefficients of the renormalised series (3.11) will be operators acting on Λ.

Consider the sum (2.20) over $i=1,\ldots,N$ appearing in (2.20). By (2.19) the action of this sum on the subspace $\Lambda_N\subset\mathbb{F}(x_1,\ldots,x_N)$ coincides with that of

$$V_1\,\gamma_1\,(1+u\,Z_1)^{-1} \tag{3.6}$$

where we set

$$V_1 \,=\, (\,t-1\,)\,\sum_{i=1}^{N}\Big(\prod_{\substack{1\leqslant l\leqslant N\\ l\neq i}} A_{il}\Big)\,\sigma_{1i}\,.$$

Here $\sigma_{11}\,=\,1$. We will demonstrate that the operator V_1 maps the subspace (3.1) to Λ_N. At the same time we will determine the inverse limit at $N\to\infty$ of the restriction of the operator V_1 to the subspace (3.1). The latter limit will be an operator $\Lambda[v]\to\Lambda[w]$ denoted simply by V. To determine this limit extend the canonical homomorphism $\Lambda\to\Lambda_N$ to a homomorphism

$$\tau_N\,:\,\Lambda[w]\to\Lambda_N\,:\,w\,\mapsto\,t^N\,.$$

Here

$$\tau_N\,:\,p_n\,\mapsto\,p_n(x_1,\dots,x_N)\quad\text{for}\quad n=1,2,\,\dots\,.$$

This definition of the homomorphism τ_N goes back to [12, Section 6]. Now define V explicitly as the unique Λ-linear operator $\Lambda[v]\to\Lambda[w]$ such that

$$V\,:\,v^n\,\mapsto\begin{cases}-\,Q_n & \text{if}\quad n>0\,,\\ w-1 & \text{if}\quad n=0\,.\end{cases}$$

Proposition. *We have a commutative diagram of $\mathbb{F}$-linear mappings*

$$\begin{array}{ccc}\Lambda[v] & \xrightarrow{\;V\;} & \Lambda[w]\\ {\scriptstyle\pi_N}\big\downarrow & & \big\downarrow{\scriptstyle\tau_N}\\ \Lambda_N^{(1)} & \xrightarrow[V_1]{} & \Lambda_N\end{array}\tag{3.7}$$

Proof. The operator $V:\Lambda[v]\to\Lambda[w]$ commutes with the multiplication by any $f\in\Lambda$. In turn, the restriction of the operator V_1 to the subspace (3.1) commutes with multiplication by $\pi_N(f)\in\Lambda_N$. So it suffices to show that the compositions $\tau_N\,V$ and $V_1\,\pi_N$ coincide on the elements $1,v,v^2,\,\dots\,\in\Lambda[v]$. Let us again use the generating series (3.4) of these elements. By applying $\tau_N\,V$ to (3.4) we get

$$\frac{1}{1-u\,v}\;\xmapsto[V]{}\;w-Q(u)\;\xmapsto[\tau_N]{}\;t^N-\prod_{i=1}^{N}\frac{1-u\,t\,x_i}{1-u\,x_i}\tag{3.8}$$

where we used (1.9). On the other hand, by applying $V_1\,\pi_N$ to (3.4) we get

$$\frac{1}{1-u\,v}\;\xmapsto[\pi_N]{}\;\frac{1}{1-u\,x_1}\;\xmapsto[V_1]{}\;\sum_{i=1}^{N}\frac{t-1}{1-u\,x_i}\prod_{\substack{1\leqslant l\leqslant N\\ l\neq i}}\frac{x_i-t\,x_l}{x_i-x_l}\;.\tag{3.9}$$

The results obtained in (3.8) and (3.9) are equal to each other. Indeed, consider them as rational functions of u and assume that $x_1,\dots,x_N\neq 0$. Then both rational functions vanish at $u=\infty$ and have poles at $u=x_1^{-1},\dots,x_N^{-1}$. These poles are simple, and the corresponding residues of two functions coincide. $\square$

By the surjectivity of π_N the last proposition implies that the operator V_1 maps the subspace (3.1) to Λ_N. Moreover, it implies that the inverse limit at $N \to \infty$ of the restriction of the operator sum (3.6) to the subspace (3.1) equals

$$V\gamma(1+uZ)^{-1} = \sum_{n=0}^{\infty} (-u)^n \, V\gamma Z^n .$$

By the definitions of Z, γ and V here for every $n \geqslant 0$ the composition $V\gamma Z^n$ is an operator $\Lambda[v] \to \Lambda[w]$. The above stated equality of the inverse limit follows from the commutativity of the diagram

$$\begin{array}{ccccccc}
\Lambda[v] & \xrightarrow{Z^n} & \Lambda[v] & \xrightarrow{\gamma} & \Lambda[v] & \xrightarrow{V} & \Lambda[w] \\
\downarrow{\scriptstyle \pi_N} & & \downarrow{\scriptstyle \pi_N} & & \downarrow{\scriptstyle \pi_N} & & \downarrow{\scriptstyle \tau_N} \\
\Lambda_N^{(1)} & \xrightarrow[Z_1^n]{} & \Lambda_N^{(1)} & \xrightarrow[\gamma_1]{} & \Lambda_N^{(1)} & \xrightarrow[V_1]{} & \Lambda_N
\end{array}$$

Here we use the commutativity of (3.2),(3.7) and that of the diagram obtained from (3.2) by replacing Z, Z_1 respectively by γ, γ_1. The commutativity of the diagram so obtained has been established as a part of our proof of Theorem 3.1.

Denote by δ the embedding of Λ to $\Lambda[v]$ as the subspace of degree zero in v. Then we have a commutative diagram

$$\begin{array}{ccc}
\Lambda & \xrightarrow{\delta} & \Lambda[v] \\
\downarrow & & \downarrow{\scriptstyle \pi_N} \\
\Lambda_N & \longrightarrow & \Lambda_N^{(1)}
\end{array}$$

where the left vertical arrow is the canonical projection. The bottom horizontal arrow is the natural embedding. It follows that the inverse limit of $I_N(u)$ equals

$$I(u) = V\gamma(1+uZ)^{-1}\delta . \tag{3.10}$$

By the above definition, every coefficient in the formal power series expansion of $I(u)$ in u is a certain operator $\Lambda \to \Lambda[w]$. Now consider the series

$$(1+u)(1+uw)^{-1}(1+uI(u)) \tag{3.11}$$

where the summand 1 in front of $uI(u)$ stands for the embedding of Λ to $\Lambda[w]$ as the subspace of degree zero in w. This should cause no confusion. In the next subsection we will show that the series (3.11) does not depend on w. Hence the coefficients of this series will be operators mapping the vector space Λ to itself.

Note that by the definition of the homomorphism τ_N and by the above given arguments, the series (3.11) is equal to the inverse limit at $N \to \infty$ of

$$(1+u)(1+ut^N)^{-1}(1+uI_N(u)) . \tag{3.12}$$

But by using the multiplicative formula (2.5), the eigenvalue of $D_N(u\,t)/D_N(u)$ on the trivial Macdonald polynomial $1 \in \Lambda_N$ corresponding to $\lambda = (0, 0, \ldots)$ is

$$(1+u)^{-1}(1+u\,t^N).$$

Hence our Theorem 2.5 implies that the eigenvalue of (3.12) on $1 \in \Lambda_N$ equals 1. By taking the limit at $N \to \infty$ the eigenvalue of (3.11) on the trivial Macdonald symmetric function $1 \in \Lambda$ also equals 1. This explains the definition of (3.11).

3.3. Reduced space. Here we will use the vector space decomposition

$$\Lambda[v] = \Lambda \oplus v\Lambda[v]. \tag{3.13}$$

The second direct summand in (3.13) will be called the *reduced space*. Relative to this decomposition the operator γ on $\Lambda[v]$ is represented by the 2×2 matrix with operator entries

$$\begin{bmatrix} 1 & 0 \\ \beta & \alpha \end{bmatrix}$$

where β denotes the composition of the restriction of γ to the first summand in (3.13) with the projection to the second summand. The map γ preserves the second summand, and α denotes the restriction of γ to it. Similarly, the operator W on $\Lambda[v]$ is represented by the 2×2 matrix with operator entries

$$\begin{bmatrix} 1 & Y \\ 0 & X \end{bmatrix}$$

where X and Y respectively denote the compositions of the restriction of W to the second summand in (3.13) with the projections to the first and to the second summands. Note that the operator $Z = W\gamma$ is then represented by the product

$$\begin{bmatrix} 1 & Y \\ 0 & X \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \beta & \alpha \end{bmatrix} = \begin{bmatrix} 1+Y\beta & Y\alpha \\ X\beta & X\alpha \end{bmatrix}. \tag{3.14}$$

By definition the operator $V : \Lambda[v] \to \Lambda[w]$ acts on the first direct summand in (3.13) as multiplication by $w-1$. The restriction of V to the second direct summand does not depend on w. Thus it maps $v\Lambda[v]$ to Λ. Moreover, by the definitions of V and W this restriction coincides with the operator $-Y$. Hence relative to (3.13) the operator V is represented by the row with operator entries

$$\begin{bmatrix} w-1\,, & -Y \end{bmatrix}.$$

Finally denote $L = \alpha X$. This is an operator on the reduced space $v\Lambda[v]$. We shall call it the *Lax operator* for the Macdonald symmetric functions in Λ. This terminology is justified by the following theorem.

Theorem. *The series* (3.11) *is equal to* $(1+u)(1+u+u\,J(u))^{-1}$ *where*

$$J(u) \;=\; Y(1+u\,L)^{-1}\beta.$$

In particular, the series (3.11) *does not depend on the variable* w.

Proof. Relative to (3.13) the operator $\delta : \Lambda \to \Lambda[v]$ is represented by the column with two operator entries

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Therefore the operator product $(1+u\,Z)^{-1}\,\delta$ appearing in the definition (3.10) of the series $I(u)$ is represented by the first column of the 2×2 matrix inverse to

$$\begin{bmatrix} 1+u+uY\beta & u\,Y\alpha \\ u\,X\beta & 1+u\,X\alpha \end{bmatrix}.$$

Here we employ the matrix representation (3.14) of the operator Z. To find that column we will use a well known formula for the inverse of a 2×2 block matrix with invertible diagonal blocks, see [1, Lemma 3.2]. The block matrix is assumed to be invertible too. The first entry of the first column that we find in this way is

$$\begin{gathered}(1+u+u\,Y\beta-u\,Y\alpha\,(1+u\,X\alpha)^{-1}\,u\,X\beta\,)^{-1} = \\ (1+u+u\,Y\beta-u^2\,Y\alpha\,X\,(1+u\,\alpha\,X)^{-1}\beta\,)^{-1} = \\ (1+u+u\,Y\,(1-u\,L\,(1+u\,L)^{-1})\,\beta\,)^{-1} = \\ (1+u+u\,Y\,(1+u\,L)^{-1}\,\beta\,)^{-1} = \\ (1+u+u\,J(u))^{-1}\,. \end{gathered} \tag{3.15}$$

The second entry of the first column of the inverse matrix that we find is then

$$\begin{aligned} &-(1+u\,X\alpha)^{-1}\,u\,X\beta\,(1+u+u\,J(u))^{-1} = \\ &\;-u\,X\,(1+u\,L)^{-1}\beta\,(1+u+u\,J(u))^{-1}\,. \end{aligned} \tag{3.16}$$

The product $V\gamma$ in (3.10) is represented by the row with operator entries

$$\begin{bmatrix} w-1\,,\,-Y \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \beta & \alpha \end{bmatrix} = \begin{bmatrix} w-1-Y\beta\,,\,-Y\alpha \end{bmatrix}.$$

The series $I(u)$ is equal to the product of this row by the column representing $(1+u\,Z)^{-1}\,\delta$. That column has the entries (3.15) and (3.16). Hence $I(u)$ equals

$$\begin{aligned} &(w-1-Y\beta)(1+u+u\,J(u))^{-1} + Y\alpha\,u\,X\,(1+u\,L)^{-1}\beta\,(1+u+u\,J(u))^{-1} \\ &\quad= (w-1-Y(1-u\,L\,(1+u\,L)^{-1})\,\beta\,)(1+u+u\,J(u))^{-1} \\ &\quad= (w-1-Y\,(1+u\,L)^{-1}\beta\,)(1+u+u\,J(u))^{-1} \\ &\quad= (w-1-J(u))(1+u+u\,J(u))^{-1}\,. \end{aligned}$$

Our theorem immediately follows from the last displayed expression for $I(u)$. □

Note that by replacing in the series $-u\,J(u)$ the variable u by $-u^{-1}$ we get the same generating series for the limits of the quantum Hamiltonians at $N\to\infty$ as was denoted in [11, Section 2] by $I(u)$. But in the present article the notation $I(u)$ was introduced in (3.10) and has a meaning different from that in [11].

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