# ON WEIGHTED TWISTED BADLY APPROXIMABLE NUMBERS

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Abstract. For any $j_1, \ldots, j_n \geq 0$ with $\sum_{i=1}^n j_i = 1$ and any $\theta \in \mathbb{R}^n$, let $\mathrm{Bad}_\theta^\times(j_1, \ldots, j_n)$ denote the set of points $\eta \in \mathbb{R}^n$ for which $\max_{1\leq i\leq n}(\|q\theta_i - \eta_i\|^{1/j_i}) > c/q$ for some positive constant $c = c(\eta)$ and all $q \in \mathbb{N}$. These sets are the 'twisted' inhomogeneous analogue of $\mathrm{Bad}(j_1, \ldots, j_n)$ in the theory of simultaneous Diophantine approximation. It has been shown that they have full Hausdorff dimension in the non-weighted setting, i.e provided that $j_i = 1/n$, and in the weighted setting when $\theta$ is chosen from $\mathrm{Bad}(j_1, \ldots, j_n)$. We generalise these results proving the full Hausdorff dimension in the weighted setting without any condition on $\theta$. Moreover, we prove $\dim(\mathrm{Bad}_\theta^\times(j_1, \ldots, j_n) \cap \mathrm{Bad}(1, 0, \ldots, 0) \cap \ldots \cap \mathrm{Bad}(0, \ldots, 0, 1)) = n$.

## 1. Introduction

The classical result due to Dirichlet: for any real number $\theta$ there exist infinitely many natural numbers $q$ such that

$$\|q\theta\| \leq q^{-1}, \tag{1}$$

where $\|\cdot\|$ denotes the distance to the nearest integer, has higher dimension generalisations. Consider any $n$-tuple of real numbers $(j_1, \ldots, j_n)$ such that

$$j_1, \ldots, j_n \geq 0 \quad \text{and} \quad \sum_{i=1}^n j_i = 1. \tag{2}$$

Then, for any vector $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$, there exist infinitely many natural numbers $q$ such that

$$\max_{1\leq i\leq n}(\|q\theta_i\|^{1/j_i}) \leq q^{-1}. \tag{3}$$

The two results above motivate the study of real numbers and real vectors $\theta \in \mathbb{R}^n$ for which the right hand side of (1) and (3) respectively

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cannot be improved by an arbitrary constant. They respectively constitute the sets Bad of badly approximable numbers and $\mathrm{Bad}(j_1,\dots,j_n)$ of $(j_1,\dots,j_n)$-badly approximable numbers. Hence

$$\mathrm{Bad}(j_1,\dots,j_n) := \left\{(\theta_1,\dots,\theta_n)\in\mathbb{R}^n \,:\, \inf_{q\in\mathbb{N}}\max_{1\le i\le n}(q^{j_i}\|q\theta_i\|)>0\right\}.$$

In the 1-dimensional case, it is well known that the set of badly approximable numbers has Lebesgue measure zero but maximal Hausdorff dimension. In the $n$-dimensional case, it is also a classical result that $\mathrm{Bad}(j_1,\dots,j_n)$ has Lebesgue measure zero, and Schmidt proved in 1966 that the particular set $\mathrm{Bad}(1/2,1/2)$ has full Hausdorff dimension. But the result of maximal dimension in the weigthed setting hasn't been proved until almost 40 years later, by Pollington and Velani [21]. In the 2-dimensional case, An showed in [1] that $\mathrm{Bad}(j_1,j_2)$ is in fact winning for the now famous Schmidt games -see [22]. Thus he provided a direct proof of a conjecture of Schmidt stating that any countable intersection of sets $\mathrm{Bad}(j_1,j_2)$ is non empty -see also [2].

Recently, interest in the size of related sets, usually referred to as the 'twists' of the sets $\mathrm{Bad}(j_1,\dots,j_n)$ has developed. The study of this twist started in the 1-dimensional setting: we fix $\theta\in\mathbb{R}$ and consider the twist of Bad:

$$\mathrm{Bad}_\theta^\times := \left\{\eta\in\mathbb{R} : \inf_{q\in\mathbb{N}} q\|q\theta-\eta\|>0\right\}.$$

The set $\mathrm{Bad}_\theta^\times$ has a palpable interpretation in terms of rotations of the unit circle. Identifying the circle with the unit interval $[0,1)$, the value $q\theta$ (modulo 1) may be thought of as the position of the origin after $q$ rotations by the angle $\theta$. If $\theta$ is rational, the rotation is periodic. If $\theta$ is irrational, a classical result of Weyl [25] implies that $q\theta$ (modulo 1) is equidistributed, so $q\theta$ visits any fixed subinterval of $[0,1)$ infinitely often. The natural question of what happens if the subinterval is allowed to shrink with time arises. Shrinking a subinterval corresponds to making its length decay according to some specified function. The set $\mathrm{Bad}_\theta^\times$ corresponds to considering, for any $\epsilon>0$, the shrinking interval $(\eta-\epsilon/q,\eta+\epsilon/q)$ centred at the point $\eta$ and where the specified function is $\epsilon/q$. Khintchine showed in [14] that

$$\|q\theta-\eta\| < \frac{1+\delta}{\sqrt{5}q} \qquad (\delta>0) \tag{4}$$

is satisfied for infinitely many integers $q$, and Theorem III in Chapter III of Cassels' book [5] shows that the right hand side of (4) cannot be improved by an arbitrary constant for every irrational $\theta$ and every real $\eta$. Again the study of the set $\mathrm{Bad}_\theta^\times$: Kim [16] proved in 2007 that it has Lebesgue measure zero, and later it was shown by Bugeaud et al [3] that it has full Hausdorff dimension. Furthermore, Tseng proved in [23] that $\mathrm{Bad}_\theta^\times$ has the stronger property of being winning for any $\theta \in \mathbb{R}$.

By generalising circle rotations to rotations on torus of higher dimensions, i.e. by considering the sequence $q\theta$ (modulo 1) in $[0,1)^n$ where $\theta = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$, we obtain the 'twists' of the sets $\mathrm{Bad}(j_1, \dots, j_n)$:
(5)

$$\mathrm{Bad}_\theta^\times(j_1, \dots, j_n) = \left\{ (\eta_1, \dots, \eta_n) \in \mathbb{R}^n : \inf_{q \in \mathbb{N}} \max_{1 \le i \le n} (q^{j_i} \|q\theta_i - \eta_i\|) > 0 \right\}.$$

In [3] Bugeaud et al proved that the non-weighted set $\mathrm{Bad}_\theta^\times(1/n, \dots$ $\dots, 1/n)$ has full Hausdorff dimension. Recently, Einsiedler and Tseng [8] extended the results [3] and [23] by showing, among other results, that $\mathrm{Bad}_\theta^\times(1/n, \dots, 1/n)$ is also winning. It was shown in [18] that such results may be obtained by classical methods developed by Khintchine [15] and V. Jarník [12,13] and discussed in Chapter V of Cassels' book [5]. Unfortunately, these methods cannot be directly extended to the weighted setting.

For the weighted setting, less has heretofore been known. Harrap did the first contribution [10] in the 2-dimensional case, by proving that $\mathrm{Bad}_\theta^\times(j_1, j_2)$ has full Hausdorff dimension provided that the fixed point $\theta \in \mathbb{R}^2$ belongs to $\mathrm{Bad}(j_1, j_2)$, which is a significantly restrictive condition. Recently, under the hypothesis $\theta \in \mathrm{Bad}(j_1, \dots, j_n)$, Harrap and Moshchevitin have extended to weighted linear forms in higher dimension and improved to winning the result in [10] (see [11]).

In this paper, we prove that the weighted set $\mathrm{Bad}_\theta^\times(j_1, \dots, j_n)$ has full Hausdorff dimension for any $\theta \in \mathbb{R}^n$. Moreover, the following theorem holds.

**Theorem 1.1.** *For any $\theta \in \mathbb{R}^n$ and all $j_1, \dots, j_n \ge 0$ with $\sum_{i=1}^n j_i = 1$,*

$$\dim(\mathrm{Bad}_\theta^\times(j_1, \dots, j_n) \cap \mathrm{Bad}(1, 0, \dots, 0) \cap \dots \cap \mathrm{Bad}(0, \dots, 0, 1)) = n.$$

The same type of theorem holds in the classical -not twisted- setting; it constitutes the work done in [21] (see Theorem 2).

Note that if $1, \theta_1, \dots, \theta_n$ are linearly dependent over $\mathbb{Z}$, then Theorem 1.1 is obvious. Indeed, in this case $\{q\theta : q \in \mathbb{Z}\}$ is restricted to a hyperplane $H$ of $\mathbb{R}^n$, so $\mathrm{Bad}_\theta^\times(j_1, \dots, j_n) \supset \mathbb{R}^n \backslash H$ is winning. Hence $\mathrm{Bad}_\theta^\times(j_1, \dots, j_n) \cap \mathrm{Bad}(1, 0, \dots, 0) \cap \dots \cap \mathrm{Bad}(0, \dots, 0, 1)$ is winning and in particular has full dimension [1]. Therefore we suppose throughout the paper that $1, \theta_1, \dots, \theta_n$ are linearly independent over $\mathbb{Z}$.

The strategy for the proof of Theorem 1.1 is as follows. We start by defining a set $\mathcal{V} \subset \mathrm{Bad}_\theta^\times(j_1, \dots, j_n)$ related to the best approximations to the fixed point $\theta \in \mathbb{R}^n$. Then we construct a Cantor-type set $K(R)$ inside $\mathcal{V} \cap \mathrm{Bad}(1, 0, \dots, 0) \cap \dots \cap \mathrm{Bad}(0, \dots, 0, 1)$. Finally we describe a probability measure supported on $K(R)$ to which we can apply the mass distribution principle and thus find a lower bound for the dimension of $K(R)$.

Best approximations are defined in Section 2. In Section 3 we define $\mathcal{V}$ and give the proof of the inclusion $\mathcal{V} \subset \mathrm{Bad}_\theta^\times(j_1, \dots, j_n)$. We construct $K(R)$ in Section 4 and describe the probability measure in Section 5. Finally we compute the lower bound for the dimension of $K(R)$ in Section 6.

In the following, $n \in \mathbb{N}$ and we fix an $n$-tuple $(j_1, \dots, j_n) \in \mathbb{R}^n$ satisfying (2) and a vector $\theta = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$ such that $1, \theta_1, \dots, \theta_n$ are linearly independent over $\mathbb{Z}$. We denote by $x \cdot y$ the scalar product of two vectors $x$ and $y$ in $\mathbb{R}^n$, and by $\|\cdot\|$ the distance to the nearest integer.

## 2. Best approximations

**Definition 2.1.** *An $n$-dimensional vector $m = (m_1, \dots, m_n) \in \mathbb{Z}^n \setminus \{0\}$ is called a best approximation to $\theta$ if for all $v \in \mathbb{Z}^n \setminus \{0, -m, m\}$ the following implication holds:*

$$\max_{1 \le i \le n} (|v_i|^{1/j_i}) \le \max_{1 \le i \le n} (|m_i|^{1/j_i}) \Longrightarrow \|v \cdot \theta\| > \|m \cdot \theta\|.$$

Note that the condition $1, \theta_1, \dots, \theta_n$ are $\mathbb{Z}$-linearly independent allows us to demand a strict inequality in the right hand side of the implication above.

Note also that when $n = 1$ the best approximations to a real number $x$ are, up to the sign, the denominators of the convergents to $x$.

Since $1, \theta_1, \dots, \theta_n$ are $\mathbb{Z}$-linearly independent, we have an infinite number of best approximations to $\theta$. They can be arranged up to the sign -so that two vectors of opposite sign do not both appear- in an infinite sequence

$$m_\nu = (m_{\nu,1}, \dots, m_{\nu,n}) \qquad \nu \ge 1, \tag{6}$$

[1]We recall that winning sets in $\mathbb{R}^n$ have maximal Hausdorff dimension, and that countable intersections of winning sets are again winning. We refer the reader to [22] for all necessary definitions and results on winning sets.

such that the values

$$M_\nu = \max_{1\le i\le n}(|m_{\nu,i}|^{1/j_i}) \tag{7}$$

form a strictly increasing sequence, and the values

$$\zeta_\nu = \|m_\nu \cdot \theta\| \tag{8}$$

form a strictly decreasing sequence. Hence each value $M_\nu$ corresponds to a single best approximation $m_\nu$. The quantity $M_\nu$ can be referred to as the 'height' of $m_\nu$.

Best approximation vectors have been introduced since a long time inside proofs in an unexplicit form. In particular, Voronoi [24] selected some points in a lattice that correspond exactly to the best approximation vectors (see also [7]). A recent survey on the topic is due to Chevallier [6]. Some important properties of the best approximation vectors are discussed in [19, 20]. The definition 2.1 of best approximation vectors is used in [11], and similar constructions were introduced in [17] or Section 2 of [4].

For each $\nu \ge 1$, it is easy to see that the region

$$\left\{(x_0,\dots,x_n)\in\mathbb{R}^{n+1} \;:\; \max_{1\le i\le n}(|x_i|^{1/j_i}) < M_{\nu+1},\ \left|x_0+\sum_{i=1}^n x_i\theta_i\right| < \zeta_\nu\right\}$$

does not contain any integer point different from 0. Since this region has volume $2^n M_{\nu+1}\zeta_\nu$ (see Lemma 4 in Appendix B of [5]), it follows from Minkowski's convex body theorem that

$$\zeta_\nu M_{\nu+1} \le 1. \tag{9}$$

**Lemma 2.2.** *For every $\nu \ge 1$, we have*

$$M_{\nu+2^{n+1}} \ge 2M_\nu.$$

**Proof.** Given $\nu \ge 1$, we must prove that we have at most $2^{n+1}$ vectors $m_{\nu+r}$ with $r \ge 0$ and $M_{\nu+r} < 2M_\nu$. The goal is to show that the 0-symmetric region

(10)

$$\left\{(x_0,\dots,x_n)\in\mathbb{R}^{n+1} :\ \max_{1\le i\le n}(|x_i|^{1/j_i}) < 2M_\nu,\ \left|x_0+\sum_{i=1}^n x_i\theta_i\right| \le \zeta_\nu\right\}$$

contains at most $2^{n+2}$ integer points other than 0. We can partition the region (10) into sets of the form

$$T(\xi) = \left\{\begin{array}{ll} (x_0,\dots,x_n)\in\mathbb{R}^{n+1} & :\ \max_{1\le i\le n}(|x_i-\xi_i|^{1/j_i}) \le M_\nu, \\ & \text{and } \left|x_0-\xi_0+\sum_{i=1}^n (x_i-\xi_i)\theta_i\right| \le \zeta_\nu \end{array}\right\},$$

with

$$\xi_i = 2kM_\nu^{j_i}, \qquad \xi_0 = -\sum_{i=1}^{n} \xi_i \theta_i \tag{11}$$

where $k \in \mathbb{Z}$ is such that $-2^{j_i-1} \le k \le 2^{j_i-1}$. Each region $T(\xi)$ is the translated by $(\xi_0, \ldots, \xi_n)$ of the set

$$\left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : \max_{1\le i\le n} (|x_i|^{1/j_i}) \le M_\nu, \ \left| x_0 + \sum_{i=1}^{n} x_i \theta_i \right| \le \zeta_\nu \right\},$$

which contains exactly three integer points: 0 and two best approximations with opposite sign. Hence each $T(\xi)$ contains at most four integer points. Since there are $2^n$ possible choices for $(\xi_0, \ldots, \xi_n)$ satisfying (11), the set (10) contains at most $2^{n+2}$ integer points.

□

## 3. The set $\mathcal{V}$ included in $\mathrm{Bad}_\theta^\times(j_1, \ldots, j_n)$

The following proposition allows us to work with a set defined by the best approximations to $\theta$ instead of working directly with $\mathrm{Bad}_\theta^\times(j_1, \ldots, j_n)$.

**Proposition 3.1.** *If $\eta \in \mathbb{R}^n$ satisfies*

$$\inf_\nu \|m_\nu \cdot \eta\| > 0, \tag{12}$$

*then $\eta \in \mathrm{Bad}_\theta^\times(j_1, \ldots, j_n)$.*

**Proof.** Let $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n$ satisfy

$$\|m_\nu \cdot \eta\| > \gamma \qquad \forall \nu \ge 1$$

for some $\gamma > 0$. For all $q \in \mathbb{N}$ and $\nu \ge 1$, we have the identity

$$m_\nu \cdot \eta \ = \ m_\nu \cdot (\eta - q\theta) + q\, m_\nu \cdot \theta,$$

from which we obtain the inequalities

$$\gamma < \|m_\nu \cdot \eta\| \le n \max_{1\le i\le n} (|m_{\nu,i}| \cdot \|\eta_i - q\theta_i\|) + q\zeta_\nu. \tag{13}$$

Since $\zeta_\nu$ is strictly decreasing and $\zeta_\nu \to 0$ as $\nu \to \infty$, there exists $\nu \ge 1$ such that

$$\frac{\gamma}{2\zeta_\nu} \le q \le \frac{\gamma}{2\zeta_{\nu+1}}. \tag{14}$$

On the one hand, from the inequalities (13) and the upper bound in (14), we deduce that

$$\max_{1\le i\le n} (\|\eta_i - q\theta_i\| \cdot |m_{\nu+1,i}|) > \frac{\gamma}{2n}. \tag{15}$$

On the other hand, from the lower bound in (14) and the inequality (9), it follows that

$$q \geq \frac{\gamma}{2} M_{\nu+1}.$$

We deduce that

$$q^{j_i} \geq c|m_{\nu+1,i}| \qquad \forall i = 1, \dots, n, \tag{16}$$

where

$$c = \min_{1 \leq i \leq n} \left( \left( \frac{\gamma}{2} \right)^{j_i} \right).$$

Finally, by combining (15) and (16), we have that

$$\max_{1 \leq i \leq n} (\|\eta_i - q\theta_i\| q^{j_i}) > \frac{\gamma c}{2n}.$$

This concludes the proof of the proposition.

□

We define the set

$$\mathcal{V} := \left\{ \eta \in \mathbb{R}^n \ : \ \inf_{\nu \geq 1} \|m_\nu \cdot \eta\| > 0 \right\}.$$

Clearly

$$\mathcal{V} \subset \mathrm{Bad}_\theta^\times(j_1, \dots, j_n). \tag{17}$$

## 4. The Cantor-type set $K(R)$

In this section we construct the Cantor-type set $K(R)$ inside $\mathrm{Bad}_\theta^\times(j_1, \dots, j_n) \cap \mathrm{Bad}(1, 0, \dots, 0) \cap \dots \cap \mathrm{Bad}(0, \dots, 0, 1)$. In order to lighten the notation, throughout this section we denote by $\mathcal{M}$ the set of best approximations in the sequence (6), and for each $m \in \mathcal{M}$, by $M_m$ the quantity defined by (7), i.e.

$$M_m = \max_{1 \leq i \leq n} (|m_i|^{1/j_i}).$$

Hence

$$\mathcal{V} = \left\{ \eta \in \mathbb{R}^n \ : \ \inf_{m \in \mathcal{M}} \|m \cdot \eta\| > 0 \right\}.$$

We define the following partition of $\mathcal{M}$:

$$\mathcal{M}_k := \left\{ m \in \mathcal{M} : R^{k-1} \leq M_m < R^k \right\} \qquad (k \geq 0). \tag{18}$$

Note that $\mathcal{M}_0 = \emptyset$. We have that $\mathcal{M} = \bigcup_{k=0}^\infty \mathcal{M}_k$.

We also need, for each $1 \leq i \leq n$, the following partitions of $\mathbb{N}$:

$$\mathcal{Q}_k^{(i)} := \left\{ q \in \mathbb{N} : R^{(k-1)j_i/2} \leq q < R^{kj_i/2} \right\} \qquad (k \geq 0) \tag{19}$$

Note that $\mathcal{Q}_0^{(i)} = \emptyset$ and for each $1 \le i \le n$, we have that $\mathbb{N} = \bigcup_{k=0}^{\infty} \mathcal{Q}_k^{(i)}$.

At the heart of the construction of $K(R)$ is constructing a collection $\mathcal{F}_k$ of hyperrectangles $H_k$ inside the hypercube $[0,1]^n$ that satisfy the following $n$ conditions:

(0) $|m \cdot \eta + p| > \epsilon \qquad \forall \eta \in H_k,\ \forall m \in \mathcal{M}_k,\ \forall p \in \mathbb{Z}$;
(1) $q|q\eta_1 - p| > \epsilon \qquad \forall \eta \in H_k,\ \forall q \in \mathcal{Q}_k^{(1)},\ \forall p \in \mathbb{Z}$;
$\vdots$
($n$) $q|q\eta_n - p| > \epsilon \qquad \forall \eta \in H_k,\ \forall q \in \mathcal{Q}_k^{(n)},\ \forall p \in \mathbb{Z}$

for some $\epsilon > 0$.

We start by constructing a collection $(\mathcal{G}_k^{(0)})_{k\ge 0}$ of hyperrectangles satisfying condition (0). This construction is done by induction. Then we define a subcollection $\mathcal{G}_k^{(1)} \subset \mathcal{G}_k^{(0)}$ of hyperrectangles that also satisfy condition (1), a subcollection $\mathcal{G}_k^{(2)} \subset \mathcal{G}_k^{(1)}$ that also satisfies condition (2), etc. This process ends with a subcollection $\mathcal{G}_k^{(n)}$ that satisfies the $n$ conditions above. We would like to quantify $\#\mathcal{G}_k^{(n)}$. We can give a lower bound, but we cannot quantify the exact cardinal. So we refine the collection $\mathcal{G}_k^{(n)}$ by choosing a right and final subcollection $\mathcal{F}_k$ that we can quantify.

Let

$$j_{\min} = \min_{1\le i\le n}(j_i), \qquad j_{\max} = \max_{1\le i\le n}(j_i).$$

Let $R > 4^{1/j_{\min}}$ and $\epsilon > 0$ be such that

$$\epsilon < \frac{1}{2R^{2j_{\max}}}. \tag{20}$$

The parameter $R$ will be chosen later to be sufficiently large in order to satisfy various conditions.

### 4.1. The collection $\mathcal{G}_k^{(0)}$.

For each $m \in \mathcal{M}$ and $p \in \mathbb{Z}$, let

$$\Delta(m,p) := \{x \in \mathbb{R}^n : |m \cdot x + p| < \epsilon\}.$$

Geometrically, $\Delta(m,p)$ is the thickening of a hyperplane of the form

$$\mathcal{L}(m,p) := \{x \in \mathbb{R}^n : m \cdot x + p = 0\} \tag{21}$$

with width $\epsilon/m_i$ in all the $x_i$-coordinate directions. The set $\mathcal{V}$ consists of points in $\mathbb{R}^n$ that avoid the thickening $\Delta(m,p)$ of each hyperplane $\mathcal{L}(m,p)$; alternatively, points in $\mathbb{R}^n$ that lie within any such neighbourhood are removed.

Next we describe the induction procedure in order to define the collection $(\mathcal{G}_k^{(0)})_{k\geq 0}$. We work within the closed hypercube $H_0 = [0,1]^n$ and set $\mathcal{G}_0^{(0)} = \{H_0\}$. For $k \geq 0$, we divide each $H_k \in \mathcal{G}_k^{(0)}$ into new hyperrectangles $H_{k+1}$ of size

$$R^{-(k+1)j_1} \times \ldots \times R^{-(k+1)j_n}.$$

Among these new hyperrectangles, we denote by $\mathcal{G}^{(0)}(H_k)$ the collection of hyperrectangles $H_{k+1} \subset H_k$ satisfying

- $G_n \subset G_{n-1}$,
- $H_{k+1} \cap \Delta(m,p) = \emptyset \qquad \forall m \in \mathcal{M}_k,\ \forall p \in \mathbb{Z}$.

We define

$$\mathcal{G}_{k+1}^{(0)} := \bigcup_{H_k \in \mathcal{G}_k^{(0)}} \mathcal{G}^{(0)}(H_k).$$

Hence $\mathcal{G}_{k+1}^{(0)}$ is nested in $\mathcal{G}_k^{(0)}$ and it is a collection of 'good' hyperrectangles with respect to all the best approximations $m$ satisfying $M_m < R^k$ and all the integers $p$. The collection $\mathcal{G}^{(0)}(H_k)$ is the collection of 'good' hyperrectangles that we obtain from the division of $H_k$.

Next we give a lower bound for $\#\mathcal{G}_k^{(0)}$. Actually, for a fixed hyperrectangle $H_k \in \mathcal{G}_k^{(0)}$, we give a lower bound for the number of hyperrectangles $H_{k+1} \in \mathcal{G}^{(0)}(H_k)$. Alternatively, we give an upper bound for the number of 'bad' hyperrectangles in $H_k$; these are the hyperrectangles $H_{k+1} \subset H_k$ that intersect the thickening $\Delta(m,p)$ of some hyperplane $\mathcal{L}(m,p)$ with $m \in \mathcal{M}_k$. Fact 1 and Fact 2 bound the number of thickenings $\Delta(m,p)$ with $m \in \mathcal{M}_k$ and $p \in \mathbb{Z}$ that intersect $H_k$. Fact 3 bounds the number of hyperrectangles $H_{k+1} \subset H_k$ that each $\Delta(m,p)$ as above intersects.

**Fact 1.** We show that for each $k \geq 1$, the set $\mathcal{M}_k$ contains at most $2^{n+1}(1+\log_2(R))$ best approximations. Let $k \geq 0$ and $r$ be the non-negative integer such that

$$2^r \leq R^{k-1} < 2^{r+1}.$$

By (18), each $m = m_\nu \in \mathcal{M}_k$ satisfies

$$M_\nu \geq 2^r. \tag{22}$$

Now Lemma 2.2 implies that

$$\begin{aligned} M_{\nu+2^{n+1}(1+\log_2(R))} &\geq 2^{1+\log_2(R)} M_\nu \\ &\geq 2^{1+\log_2(R)+r} \\ &> R^k. \end{aligned}$$

Therefore, there are at most $2^{n+1}(1+\log_2(R))$ best approximations in $\mathcal{M}_k$.

**Fact 2.** Fix $m \in \mathcal{M}_k$. We show that there are at most $2^n n$ thickenings $\Delta(m,p)$ that intersect $H_k$. Indeed, suppose that two different thickenings $\Delta(m,p)$ and $\Delta(m,p')$ intersect the same edge of $H_k$. This edge of $H_k$ is a segment of a line which is parallel to an $x_l$-axis. Let $P=(y_1,\ldots,y_n)$ and $P'=(y'_1,\ldots,y'_n)$ denote the points of intersection of this line parallel to the $x_l$-axis with $\mathcal{L}(m,p)$ and $\mathcal{L}(m,p')$ respectively. The fact that $P$ and $P'$ respectively belong to $\mathcal{L}(m,p)$ and $\mathcal{L}(m,p')$ is described by the equations

$$m\cdot y+p=0, \qquad m\cdot y'+p'=0. \tag{23}$$

The fact that $P$ and $P'$ both belong to a line parallel to the $x_l$-axis, implies that $y_i=y'_i\ \forall i\neq l$. Hence, by substracting the second equation in (23) to the first one, we have that

$$|y_l-y'_l|-\frac{2\epsilon}{|m_l|}\geq\frac{|p-p'|}{|m_l|}-\frac{2\epsilon}{|m_l|}>\frac{1}{R^{kj_l}}-\frac{1}{2R^{kj_l}}=\frac{1}{2}R^{-kj_l}. \tag{24}$$

Since the size of $H_k$ in the $x_l$-direction is $R^{-kj_l}$, the inequality (24) implies that there are not more than two thickenings intersecting $H_k$. Thus the number of thickenings $\Delta(m,p)$ that intersect $H_k$ is at most twice the number of edges of $H_k$, and this is $2^n n$.

**Fact 3.** Given a thickening $\Delta(m,p)$, we give an upper bound for the number of hyperrectangles $H_{k+1}\subset H_k$ that intersect $\Delta(m,p)$. Fix $m\in\mathcal{M}_k$ and $p\in\mathbb{Z}$. Denote by $l$ the index such that $M_m=|m_l|^{1/j_l}$. The inequalities

$$\frac{\epsilon}{|m_l|}<\frac{1}{2R^{(k+1)j_l}} \qquad \text{and} \qquad \frac{\epsilon}{|m_l|}\leq\frac{\epsilon}{|m_i|} \qquad (1\leq i\leq n)$$

imply that $\Delta(m,p)$ intersects at most two hyperrectangles on each line of hyperrectangles along the $x_l$-direction. Since there are $[R^{j_1}]\times\ldots\times\widehat{[R^{j_l}]}\times\ldots\times[R^{j_n}]$ lines of hyperrectangles along the $x_l$-direction, the thickening $\Delta(m,p)$ intersects at most

$$2[R^{j_1}]\times\ldots\times\widehat{[R^{j_l}]}\times\ldots\times[R^{j_n}]\leq 2R^{1-j_l}$$

hyperrectangles $H_{k+1}\subset H_k$. Hence we conclude that $\Delta(m,p)$ intersects at most $2R^{1-j_l}$ hyperrectangles $H_{k+1}\subset H_k$.

**Conclusion.** There are at most $[2^{n+1}n(1+\log_2(R))R^{1-j_{\min}}]$ hyperrectangles $H_{k+1}\subset H_k$ that intersect some $\Delta(m,p)$ with $m\in\mathcal{M}_k$,

$p \in \mathbb{Z}$. Hence

$$\#\mathcal{G}^{(0)}(H_k) \geq [R] - [2^{n+1}n(1 + \log_2(R))R^{1-j_{\min}}].$$

4.2. **The subcollections $\mathcal{G}_k^{(i)}$.** For each $q \in \mathbb{N}$ and $p \in \mathbb{Z}$, consider the sets

$$\Gamma_i(q, p) := \{x \in \mathbb{R}^n : q|qx_i - p| < \epsilon\} \qquad (1 \leq i \leq n). \tag{25}$$

Geometrically, each $\Gamma_i(q, p)$ is a thickening of a hyperplane described by the equation $x_i = p/q$ with width $\epsilon/q^2$ in the $x_i$-coordinate direction.

We construct a tower of subcollections

$$\mathcal{G}_k^{(n)} \subset \ldots \subset \mathcal{G}_k^{(1)} \subset \mathcal{G}_k^{(0)},$$

where each $\mathcal{G}_k^{(i)}$ consists of hyperrectangles in $\mathcal{G}_k^{(i-1)}$ which points avoid each thickening $\Gamma_i(q, p)$ for $q \in \mathcal{Q}_k^{(i)}$. More precisely, for $1 \leq i \leq n$, we form $\mathcal{G}_k^{(i)}$ by letting

$$\mathcal{G}^{(i)}(H_k) := \left\{H_{k+1} \in \mathcal{G}^{(i-1)}(H_k) : H_{k+1} \cap \Gamma_i(q, p) = \emptyset \ \forall q \in \mathcal{Q}_k^{(i)}\right\}$$

and

$$\mathcal{G}_{k+1}^{(i)} := \bigcup_{H_k \in \mathcal{G}_k^{(i-1)}} \mathcal{G}^{(i)}(H_k).$$

Clearly the hyperrectangles in $\mathcal{G}_{k+1}^{(i)}$ satisfy the conditions (0),(1),...,($i$), so $\mathcal{G}_k^{(n)}$ satisfies the $n$ conditions (0),...,($n$).

Next, for each $1 \leq i \leq n$ and $H_k \in \mathcal{G}_k^{(i-1)}$, we give a lower bound of $\#\mathcal{G}^{(i)}(H_k)$.

Fix $i \in \mathbb{N}$, $i \leq n$ and $H_k \in \mathcal{G}_k^{(i-1)}$. Suppose that there are two pairs $(q, p)$ and $(q', p')$ in $\mathcal{Q}_k^{(i)} \times \mathbb{Z}$ such that

$$H_k \cap \Gamma_i(q, p) \neq \emptyset, \qquad H_k \cap \Gamma_i(q', p') \neq \emptyset.$$

In other words, suppose there exist $\eta, \eta'$ in $H_k$ such that

$$q|q\eta_i - p| < \epsilon, \qquad q|q'\eta_i' - p'| < \epsilon. \tag{26}$$

Then, by (19) and (20), we have

$$\left|\frac{p}{q} - \frac{p'}{q'}\right| - \frac{\epsilon}{q^2} - \frac{\epsilon}{q'^2} \geq \frac{1}{qq'} - \frac{\epsilon}{q^2} - \frac{\epsilon}{q'^2} > \frac{1}{R^{kj_i}} - \frac{1}{2R^{kj_i}} = \frac{1}{2}R^{-kj_i}. \tag{27}$$

Since the size of $H_k$ in the $x_i$-direction is $R^{-kj_i}$, the inequality (27) implies that at most two thickenings of the form (25) can intersect $H_k$.

Now, from (26), (19) and (20), it follows that if $\eta \in \Gamma_i(q,p)$, then

$$\left|\eta_i - \frac{p}{q}\right| < \frac{\epsilon}{q^2} < \frac{1}{2}R^{-kj_i}.$$

The inequality above implies that each thickening $\Gamma_i(q,p)$ intersects at most

$$2[R^{j_1}] \times \ldots \times \widehat{[R^{j_i}]} \times \ldots \times [R^{j_n}] \leq 2R^{1-j_i}$$

hyperrectangles $H_{k+1} \subset H_k$.

Therefore, there are at most $[4R^{1-j_{\min}}]$ hyperrectangles $H_{k+1} \subset H_k$ that do not satisfy condition (i). Hence

$$\#\mathcal{G}^{(i)}(H_k) \geq [R] - [2^{n+1}n(1+\log_2(R))R^{1-j_{\min}}] - [4iR^{1-j_{\min}}]. \tag{28}$$

4.3. **The right subcollection $\mathcal{F}_k$.** We choose a subcollection of $\mathcal{G}_k^{(n)}$ that we can exactly quantify in the following way. Let $\mathcal{F}_0 := \mathcal{G}_0^{(0)}$. Choose $R$ sufficiently large so that $[R - 2^{n+1}n(1+\log_2(R))R^{1-j_{\min}} - 4nR^{1-j_{\min}}] > 1$. For $k \geq 0$, for each $H_k \in \mathcal{F}_k$, we choose exactly $[R - 2^{n+1}n(1+\log_2(R))R^{1-j_{\min}} - 4nR^{1-j_{\min}}]$ hyperrectangles from the collection $\mathcal{G}^{(n)}(H_k)$; denote this collection by $\mathcal{F}(H_k)$. Trivially,

$$\#\mathcal{F}(H_k) = [R - 2^{n+1}n(1+\log_2(R))R^{1-j_{\min}} - 4nR^{1-j_{\min}}] > 1, \tag{29}$$

so each hyperrectangle $H_k \in \mathcal{F}_k$ gives rise to exactly the same number of hyperrectangles $H_{k+1}$ in $\mathcal{F}(H_k)$. Finally, define

$$\mathcal{F}_{k+1} := \bigcup_{H_k \in \mathcal{F}_k} \mathcal{F}(H_k).$$

This completes the construction of the Cantor-type set

$$K(R) := \bigcap_{k=0}^{\infty} \mathcal{F}_k.$$

By construction, we have $K(R) \subset \mathcal{V} \cap \mathrm{Bad}(1,0\ldots,0) \cap \ldots \cap \mathrm{Bad}(0,\ldots,0,1)$. Moreover, in view of (29), we have

$$\#\mathcal{F}_{k+1} = \#\mathcal{F}_k \, \#\mathcal{F}(H_k) \tag{30}$$

$$= [R - 2^{n+1}n(1+\log_2(R))R^{1-j_{\min}} - 4nR^{1-j_{\min}}]^{k+1}. \tag{31}$$

## 5. The measure $\mu$ on $K(R)$

We now describe a probablity measure $\mu$ supported on the Cantor-type set $K(R)$ constructed in the previous section. The measure we define is analogous to the probability measure used in [21] and [2] on a Cantor-type set of $\mathbb{R}^2$. For any hyperrectangle $H_k \in \mathcal{F}_k$ we attach a weight $\mu(H_k)$ which is defined recursively as follows: for $k = 0$,

$$\mu(H_0) = \frac{1}{\#\mathcal{F}_0} = 1$$

and for $k \geq 1$,

$$\mu(H_k) = \frac{1}{\#\mathcal{F}(H_{k-1})}\mu(H_{k-1}) \qquad (H_k \in \mathcal{F}(H_{k-1})).$$

This procedure defines inductively a mass on any hyperrectangle used in the construction of $K(R)$. Moreover, $\mu$ can be further extended to all Borel subsets $X$ of $\mathbb{R}^n$, so that $\mu$ actually defines a measure supported on $K(R)$, by letting

$$\mu(X) = \inf \sum_{H \in \mathcal{C}} \mu(H)$$

where the infimum is taken over all coverings $\mathcal{C}$ of $X$ by rectangles $H \in \{\mathcal{F}_k : k \geq 0\}$. For further details, see [9], Proposition 1.7.

Notice that, in view of (30), we have

$$\mu(H_k) = \frac{1}{\#\mathcal{F}_k} \qquad (k \geq 0).$$

A classical method for obtaining a lower bound for the Hausdorff dimension of an arbitrary set is the following mass distribution principle (see [9] p. 55).

**Lemma 5.1** (mass distribution principle)**.** *Let $\delta$ be a probability measure supported on a subset $X$ of $\mathbb{R}^n$. Suppose there are positive constants $c, s$ and $l_0$ such that*

$$\delta(S) \leq cl^s \tag{32}$$

*for any hypercube $S \subset \mathbb{R}^n$ with side length $l \leq l_0$. Then $\dim(X) \geq s$.*

The goal in the next section is to prove that there exist constants $c$ and $l_0$ satisfying (32) with $\delta = \mu$, $X = K(R)$ and $s = n - \lambda(R)$, where $\lambda(R) \to 0$ as $R \to \infty$. Then from the mass distribution principle it will follow that $\dim(K(R)) = n$.

## 6. A lower bound for $\dim(K(R))$

Recall that

$$j_{\min} = \min_{1\le i\le n}(j_i).$$

Let $k_0$ be a positive integer such that

$$R^{-kj_i} < R^{-(k+1)j_{\min}} \qquad \forall j_i \neq j_{\min} \text{ and } k \geq k_0. \tag{33}$$

Consider an arbitrary hypercube $S$ of side length $l \leq l_0$ where $l_0$ satisfies

$$l_0 < R^{-(k_0+1)j_{\min}} \tag{34}$$

together with a second inequality to be determined later. We can choose $k \geq k_0$ so that

$$R^{-(k+1)j_{\min}} < l \leq R^{-kj_{\min}}. \tag{35}$$

From the inequality (33) it follows that

$$l > R^{-kj_i} \qquad \forall j_i \neq j_{\min}. \tag{36}$$

Then it is easy to see that $S$ intersects at most $2^n l^{n-1}\prod_{j_i\neq j_{\min}} R^{kj_i}$ hyperrectangles $H_k \in \mathcal{F}_k$, so

$$\mu(S) \leq 2^n l^{n-1} \prod_{j_i\neq j_{\min}} R^{kj_i}\mu(H_k) = 2^n l^{n-1} R^{k-kj_{\min}} \frac{1}{\#\mathcal{F}_k}.$$

Since $R^{(k+1)j_{\min}} > l^{-1}$ (see (35)), we have that

$$\begin{aligned}\mu(S) &\leq 2^n l^n R^{j_{\min}} R^k \frac{1}{\#\mathcal{F}_k}\\ &\leq 2^n l^n R^{j_{\min}}(1-4nR^{-j_{\min}}(2^{n-1}(1+\log_2(R))+1))^{-k}\end{aligned}$$

by applying (30). We want to choose $k$ and $\lambda(R)$ so that

$$R^{j_{\min}}(1-4nR^{-j_{\min}}(2^{n-1}(1+\log_2(R))+1))^{-k} \leq R^{kj_{\min}\lambda(R)}. \tag{37}$$

Remember we mentioned in Section 3 that later we would choose the parameter $R$ big enough so that it satisfies various conditions. We choose $R$ so that

$$4nR^{-j_{\min}}(2^{n-1}(1+\log_2(R))+1) \leq \frac{1}{2}.$$

Then, on taking

$$k \geq \log(R) \quad \text{and} \quad \lambda(R) = \frac{1+\log(2)}{j_{\min}\log(R)},$$

we have that

$$\begin{aligned} j_{\min}\log(R) - k\log(1 - 4nR^{-j_{\min}}(2^{n-1}(1+\log_2(R)) + 1)) \\ &\leq j_{\min}\log(R) + k\log(2) \\ &\leq k(1+\log(2)) \\ &= kj_{\min}\lambda(R)\log(R), \end{aligned}$$

and so the inequality (37) is satisfied. Since $R^{kj_{\min}} \leq l^{-1}$ (see (35)), it follows that

$$\mu(S) \leq 2^n l^{n-\lambda(R)}.$$

Finally, by applying the mass distribution principle we obtain

$$\dim K(R) \geq n - \lambda(R) \to n \qquad \text{as } R \to \infty.$$

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