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ON WEIGHTED TWISTED BADLY APPROXIMABLE NUMBERS

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ABSTRACT. For any $j_1, \ldots, j_n \geq 0$ with $\sum_{i=1}^{n} j_i = 1$ and any $\theta \in \mathbb{R}^n$, let $\text{Bad}_\theta^\times (j_1, \ldots, j_n)$ denote the set of points $\eta \in \mathbb{R}^n$ for which $\max_{1 \leq i \leq n} (\|q\theta_i - \eta_i\|^{1/j_i}) > c/q$ for some positive constant $c = c(\eta)$ and all $q \in \mathbb{N}$. These sets are the ‘twisted’ inhomogeneous analogue of $\text{Bad}(j_1, \ldots, j_n)$ in the theory of simultaneous Diophantine approximation. It has been shown that they have full Hausdorff dimension in the non-weighted setting, i.e provided that $j_i = 1/n$, and in the weighted setting when $\theta$ is chosen from $\text{Bad}(j_1, \ldots, j_n)$. We generalise these results proving the full Hausdorff dimension in the weighted setting without any condition on $\theta$. Moreover, we prove $\dim(\text{Bad}_\theta^\times (j_1, \ldots, j_n) \cap \text{Bad}(1, 0, \ldots, 0) \cap \ldots \cap \text{Bad}(0, \ldots, 0, 1)) = n$.

1. Introduction

The classical result due to Dirichlet: for any real number $\theta$ there exist infinitely many natural numbers $q$ such that

(1) $\|q\theta\| \leq q^{-1},$

where $\|\cdot\|$ denotes the distance to the nearest integer, has higher dimension generalisations. Consider any $n$-tuple of real numbers $(j_1, \ldots, j_n)$ such that

(2) $j_1, \ldots, j_n \geq 0$ and $\sum_{i=1}^{n} j_i = 1.$

Then, for any vector $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$, there exist infinitely many natural numbers $q$ such that

(3) $\max_{1 \leq i \leq n} (\|q\theta_i\|^{1/j_i}) \leq q^{-1}.$

The two results above motivate the study of real numbers and real vectors $\theta \in \mathbb{R}^n$ for which the right hand side of (1) and (3) respectively

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cannot be improved by an arbitrary constant. They respectively constitute the sets \( \text{Bad} \) of badly approximable numbers and \( \text{Bad}(j_1, \ldots, j_n) \) of \((j_1, \ldots, j_n)\)-badly approximable numbers. Hence

\[
\text{Bad}(j_1, \ldots, j_n) := \left\{ (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n : \inf_{q \in \mathbb{N}} \max_{1 \leq i \leq n} (q^{j_i} \|q\theta_i\|) > 0 \right\}.
\]

In the 1-dimensional case, it is well known that the set of badly approximable numbers has Lebesgue measure zero but maximal Hausdorff dimension. In the \( n \)-dimensional case, it is also a classical result that \( \text{Bad}(j_1, \ldots, j_n) \) has Lebesgue measure zero, and Schmidt proved in 1966 that the particular set \( \text{Bad}(1/2, 1/2) \) has full Hausdorff dimension. But the result of maximal dimension in the weighted setting hasn’t been proved until almost 40 years later, by Pollington and Velani \[21\]. In the 2-dimensional case, An showed in \[1\] that \( \text{Bad}(j_1, j_2) \) is in fact winning for the now famous Schmidt games -see \[22\]. Thus he provided a direct proof of a conjecture of Schmidt stating that any countable intersection of sets \( \text{Bad}(j_1, j_2) \) is non empty -see also \[2\].

Recently, interest in the size of related sets, usually referred to as the ‘twists’ of the sets \( \text{Bad}(j_1, \ldots, j_n) \) has developed. The study of this twist started in the 1-dimensional setting: we fix \( \theta \in \mathbb{R} \) and consider the twist of \( \text{Bad} \):

\[
\text{Bad}_\theta^\times := \left\{ \eta \in \mathbb{R} : \inf_{q \in \mathbb{N}} q \|q\theta - \eta\| > 0 \right\}.
\]

The set \( \text{Bad}_\theta^\times \) has a palpable interpretation in terms of rotations of the unit circle. Identifying the circle with the unit interval \([0, 1)\), the value \( q\theta \) (modulo 1) may be thought of as the position of the origin after \( q \) rotations by the angle \( \theta \). If \( \theta \) is rational, the rotation is periodic. If \( \theta \) is irrational, a classical result of Weyl \[25\] implies that \( q\theta \) (modulo 1) is equidistributed, so \( q\theta \) visits any fixed subinterval of \([0, 1)\) infinitely often. The natural question of what happens if the subinterval is allowed to shrink with time arises. Shrinking a subinterval corresponds to making its length decay according to some specified function. The set \( \text{Bad}_\theta^\times \) corresponds to considering, for any \( \epsilon > 0 \), the shrinking interval \((\eta - \epsilon/q, \eta + \epsilon/q)\) centred at the point \( \eta \) and where the specified function is \( \epsilon/q \). Khintchine showed in \[14\] that

\[
(4) \quad \|q\theta - \eta\| < \frac{1 + \delta}{\sqrt{5}q} \quad (\delta > 0)
\]

is satisfied for infinitely many integers \( q \), and Theorem III in Chapter III of Cassels’ book \[5\] shows that the right hand side of \( (4) \) cannot be improved by an arbitrary constant for every irrational \( \theta \) and every
real \( \eta \). Again the study of the set \( \text{Bad}_\theta^\times \): Kim [16] proved in 2007 that it has Lebesgue measure zero, and later it was shown by Bugeaud et al [3] that it has full Hausdorff dimension. Furthermore, Tseng proved in [23] that \( \text{Bad}_\theta^\times \) has the stronger property of being winning for any \( \theta \in \mathbb{R} \).

By generalising circle rotations to rotations on torus of higher dimensions, i.e. by considering the sequence \( q\theta \) (modulo 1) in \([0,1)^n\) where \( \theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n \), we obtain the ‘twists’ of the sets \( \text{Bad}(j_1, \ldots, j_n) \):

\[
\text{Bad}_\theta(j_1, \ldots, j_n) = \left\{ (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n : \inf_{q \in \mathbb{N}} \max_{1 \leq i \leq n} (q^j \| q\theta_i - \eta_i \|) > 0 \right\}.
\]

In [3] Bugeaud et al proved that the non-weighted set \( \text{Bad}_\theta (1/n, \ldots, 1/n) \) has full Hausdorff dimension. Recently, Einsiedler and Tseng [8] extended the results [3] and [23] by showing, among other results, that \( \text{Bad}_\theta (1/n, \ldots, 1/n) \) is also winning. It was shown in [18] that such results may be obtained by classical methods developed by Khintchine [15] and V. Jarník [12,13] and discussed in Chapter V of Cassels’ book [5]. Unfortunately, these methods cannot be directly extended to the weighted setting.

For the weighted setting, less has heretofore been known. Harrap did the first contribution [10] in the 2-dimensional case, by proving that \( \text{Bad}_\theta(j_1, j_2) \) has full Hausdorff dimension provided that the fixed point \( \theta \in \mathbb{R}^2 \) belongs to \( \text{Bad}(j_1, j_2) \), which is a significantly restrictive condition. Recently, under the hypothesis \( \theta \in \text{Bad}(j_1, \ldots, j_n) \), Harrap and Moshchevitin have extended to weighted linear forms in higher dimension and improved to winning the result in [10] (see [11]).

In this paper, we prove that the weighted set \( \text{Bad}_\theta^\times(j_1, \ldots, j_n) \) has full Hausdorff dimension for any \( \theta \in \mathbb{R}^n \). Moreover, the following theorem holds.

**Theorem 1.1.** For any \( \theta \in \mathbb{R}^n \) and all \( j_1, \ldots, j_n \geq 0 \) with \( \sum_{i=1}^n j_i = 1 \),

\[
\dim(\text{Bad}_\theta^\times(j_1, \ldots, j_n) \cap \text{Bad}(1, 0, \ldots, 0) \cap \ldots \cap \text{Bad}(0, \ldots, 0, 1)) = n.
\]

The same type of theorem holds in the classical -not twisted- setting; it constitutes the work done in [21] (see Theorem 2).

Note that if \( 1, \theta_1, \ldots, \theta_n \) are linearly dependent over \( \mathbb{Z} \), then Theorem 1.1 is obvious. Indeed, in this case \( \{q\theta : q \in \mathbb{Z}\} \) is restricted to a hyperplane \( H \) of \( \mathbb{R}^n \), so \( \text{Bad}_\theta^\times(j_1, \ldots, j_n) \supset \mathbb{R}^n \setminus H \) is winning. Hence \( \text{Bad}_\theta^\times(j_1, \ldots, j_n) \cap \text{Bad}(1, 0, \ldots, 0) \cap \ldots \cap \text{Bad}(0, \ldots, 0, 1) \) is winning and
in particular has full dimension. Therefore we suppose throughout the paper that \(1, \theta_1, \ldots, \theta_n\) are linearly independent over \(\mathbb{Z}\).

The strategy for the proof of Theorem 1.1 is as follows. We start by defining a set \(\mathcal{V} \subset \text{Bad}_n^\infty(j_1, \ldots, j_n)\) related to the best approximations to the fixed point \(\theta \in \mathbb{R}^n\). Then we construct a Cantor-type set \(K(R)\) inside \(\mathcal{V} \cap \text{Bad}(1, 0, \ldots, 0) \cap \ldots \cap \text{Bad}(0, \ldots, 0, 1)\). Finally we describe a probability measure supported on \(K(R)\) to which we can apply the mass distribution principle and thus find a lower bound for the dimension of \(K(R)\).

Best approximations are defined in Section 2. In Section 3 we define \(\mathcal{V}\) and give the proof of the inclusion \(\mathcal{V} \subset \text{Bad}_n^\infty(j_1, \ldots, j_n)\). We construct \(K(R)\) in Section 4 and describe the probability measure in Section 5. Finally we compute the lower bound for the dimension of \(K(R)\) in Section 6.

In the following, \(n \in \mathbb{N}\) and we fix an \(n\)-tuple \((j_1, \ldots, j_n) \in \mathbb{R}^n\) satisfying (2) and a vector \(\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n\) such that \(1, \theta_1, \ldots, \theta_n\) are linearly independent over \(\mathbb{Z}\). We denote by \(x \cdot y\) the scalar product of two vectors \(x\) and \(y\) in \(\mathbb{R}^n\), and by \(|\cdot|\) the distance to the nearest integer.

## 2. Best approximations

**Definition 2.1.** An \(n\)-dimensional vector \(m = (m_1, \ldots, m_n) \in \mathbb{Z}^n\setminus\{0\}\) is called a best approximation to \(\theta\) if for all \(v \in \mathbb{Z}^n\setminus\{0, -m, m\}\) the following implication holds:

\[
\max_{1 \leq i \leq n}(|v_i|^{1/j_i}) \leq \max_{1 \leq i \leq n}(|m_i|^{1/j_i}) \implies \|v \cdot \theta\| > \|m \cdot \theta\|.
\]

Note that the condition \(1, \theta_1, \ldots, \theta_n\) are \(\mathbb{Z}\)-linearly independent allows us to demand a strict inequality in the right hand side of the implication above.

Note also that when \(n = 1\) the best approximations to a real number \(x\) are, up to the sign, the denominators of the convergents to \(x\).

Since \(1, \theta_1, \ldots, \theta_n\) are \(\mathbb{Z}\)-linearly independent, we have an infinite number of best approximations to \(\theta\). They can be arranged up to the sign -so that two vectors of opposite sign do not both appear- in an infinite sequence

\[
m_\nu = (m_{\nu,1}, \ldots, m_{\nu,n}) \quad \nu \geq 1,
\]

\(^1\)We recall that winning sets in \(\mathbb{R}^n\) have maximal Hausdorff dimension, and that countable intersections of winning sets are again winning. We refer the reader to [22] for all necessary definitions and results on winning sets.
such that the values
\[ M_\nu = \max_{1 \leq i \leq n} (|m_{\nu,i}|^{1/j_i}) \]
form a strictly increasing sequence, and the values
\[ \zeta_\nu = \|m_\nu \cdot \theta\| \]
form a strictly decreasing sequence. Hence each value \( M_\nu \) corresponds to a single best approximation \( m_\nu \). The quantity \( M_\nu \) can be referred to as the ‘height’ of \( m_\nu \).

Best approximation vectors have been introduced since a long time inside proofs in an unexplicit form. In particular, Voronoi [24] selected some points in a lattice that correspond exactly to the best approximation vectors (see also [7]). A recent survey on the topic is due to Chevallier [6]. Some important properties of the best approximation vectors are discussed in [19, 20]. The definition 2.1 of best approximation vectors is used in [11], and similar constructions were introduced in [17] or Section 2 of [4].

For each \( \nu \geq 1 \), it is easy to see that the region
\[ \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : \max_{1 \leq i \leq n} (|x_i|^{1/j_i}) < M_{\nu+1}, \ |x_0 + \sum_{i=1}^{n} x_i \theta_i| < \zeta_\nu \right\} \]
does not contain any integer point different from 0. Since this region has volume \( 2^n M_{\nu+1} \zeta_\nu \) (see Lemma 4 in Appendix B of [5]), it follows from Minkowski’s convex body theorem that
\[ \zeta_\nu M_{\nu+1} \leq 1. \]

**Lemma 2.2.** For every \( \nu \geq 1 \), we have
\[ M_{\nu+2n+1} \geq 2M_\nu. \]

**Proof.** Given \( \nu \geq 1 \), we must prove that we have at most \( 2^{n+1} \) vectors \( m_{\nu+r} \) with \( r \geq 0 \) and \( M_{\nu+r} < 2M_\nu \). The goal is to show that the 0-symmetric region
\[ \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : \max_{1 \leq i \leq n} (|x_i|^{1/j_i}) < 2M_\nu, \ |x_0 + \sum_{i=1}^{n} x_i \theta_i| \leq \zeta_\nu \right\} \]
contains at most \( 2^{n+2} \) integer points other than 0. We can partition the region (11) into sets of the form
\[ T(\xi) = \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : \max_{1 \leq i \leq n} (|x_i - \xi_i|^{1/j_i}) \leq M_\nu, \quad \text{and} \quad |x_0 - \xi_0 + \sum_{i=1}^{n} (x_i - \xi_i) \theta_i| \leq \zeta_\nu \right\} , \]
with

\[(11) \quad \xi_i = 2kM^j_i, \quad \xi_0 = -\sum_{i=1}^{n} \xi_i \theta_i\]

where \(k \in \mathbb{Z}\) is such that \(-2^{j_i-1} \leq k \leq 2^{j_i-1}\). Each region \(T(\xi)\) is the translated by \((\xi_0, \ldots, \xi_n)\) of the set

\[
\left\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : \max_{1 \leq i \leq n} |x_i|^{1/j_i} \leq M_\nu, \quad \left|x_0 + \sum_{i=1}^{n} x_i \theta_i\right| \leq \zeta_\nu\right\},
\]

which contains exactly three integer points: 0 and two best approximations with opposite sign. Hence each \(T(\xi)\) contains at most four integer points. Since there are \(2^n\) possible choices for \((\xi_0, \ldots, \xi_n)\) satisfying \((11)\), the set \((10)\) contains at most \(2^{n+2}\) integer points.

\[\square\]

3. The set \(V\) included in \(\text{Bad}_\theta^\chi(j_1, \ldots, j_n)\)

The following proposition allows us to work with a set defined by the best approximations to \(\theta\) instead of working directly with \(\text{Bad}_\theta^\chi(j_1, \ldots, j_n)\).

**Proposition 3.1.** If \(\eta \in \mathbb{R}^n\) satisfies

\[(12) \quad \inf_\nu \|m_\nu \cdot \eta\| > 0,\]

then \(\eta \in \text{Bad}_\theta^\chi(j_1, \ldots, j_n)\).

**Proof.** Let \(\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n\) satisfy

\[\|m_\nu \cdot \eta\| > \gamma \quad \forall \nu \geq 1\]

for some \(\gamma > 0\). For all \(q \in \mathbb{N}\) and \(\nu \geq 1\), we have the identity

\[m_\nu \cdot \eta = m_\nu \cdot (\eta - q\theta) + q m_\nu \cdot \theta,\]

from which we obtain the inequalities

\[(13) \quad \gamma < \|m_\nu \cdot \eta\| \leq n \max_{1 \leq i \leq n} (|m_{\nu,i}| \cdot \|\eta_i - q\theta_i\|) + q \zeta_\nu.\]

Since \(\zeta_\nu\) is strictly decreasing and \(\zeta_\nu \to 0\) as \(\nu \to \infty\), there exists \(\nu \geq 1\) such that

\[(14) \quad \frac{\gamma}{2\zeta_\nu} \leq q \leq \frac{\gamma}{2\zeta_{\nu+1}}.\]

On the one hand, from the inequalities \((13)\) and the upper bound in \((14)\), we deduce that

\[(15) \quad \max_{1 \leq i \leq n} (\|\eta_i - q\theta_i\| \cdot |m_{\nu+1,i}|) > \frac{\gamma}{2n}.\]
On the other hand, from the lower bound in (14) and the inequality (9), it follows that
\[ q \geq \frac{\gamma}{2} M_{\nu+1}. \]
We deduce that
\[ q^i \geq c|m_{\nu+1,i}| \quad \forall i = 1, \ldots, n, \]
where
\[ c = \min_{1 \leq i \leq n} \left( \left( \frac{\gamma}{2} \right)^{j_i} \right). \]
Finally, by combining (15) and (16), we have that
\[ \max_{1 \leq i \leq n} (||\eta_i - q\theta_i||q^i) > \frac{\gamma c}{2n}. \]
This concludes the proof of the proposition.

We define the set
\[ V := \left\{ \eta \in \mathbb{R}^n : \inf_{\nu \geq 1} \|m_\nu \cdot \eta\| > 0 \right\}. \]
Clearly
\[ V \subset \text{Bad}^x(j_1, \ldots, j_n). \]

4. The Cantor-type set \( K(R) \)

In this section we construct the Cantor-type set \( K(R) \) inside \( \text{Bad}^x(j_1, \ldots, j_n) \cap \text{Bad}(1,0, \ldots, 0) \cap \ldots \cap \text{Bad}(0, \ldots, 0, 1) \). In order to lighten the notation, throughout this section we denote by \( \mathcal{M} \) the set of best approximations in the sequence (6), and for each \( m \in \mathcal{M} \), by \( M_m \) the quantity defined by (7), i.e.
\[ M_m = \max_{1 \leq i \leq n} (|m_i|^{1/j_i}). \]
Hence
\[ V = \left\{ \eta \in \mathbb{R}^n : \inf_{m \in \mathcal{M}} \|m \cdot \eta\| > 0 \right\}. \]
We define the following partition of \( \mathcal{M} \):
\[ \mathcal{M}_k := \left\{ m \in \mathcal{M} : R^{k-1} \leq M_m < R^k \right\} \quad (k \geq 0). \]
Note that \( \mathcal{M}_0 = \emptyset \). We have that \( \mathcal{M} = \bigcup_{k=0}^{\infty} \mathcal{M}_k \).

We also need, for each \( 1 \leq i \leq n \), the following partitions of \( \mathbb{N} \):
\[ Q_k^{(i)} := \left\{ q \in \mathbb{N} : R^{(k-1)j_i/2} \leq q < R^{k j_i/2} \right\} \quad (k \geq 0) \]
Note that $Q_0^{(i)} = \emptyset$ and for each $1 \leq i \leq n$, we have that $\mathbb{N} = \bigcup_{k=0}^{\infty} Q_k^{(i)}$.

At the heart of the construction of $K(R)$ is constructing a collection $F_k$ of hyperrectangles $H_k$ inside the hypercube $[0,1]^n$ that satisfy the following $n$ conditions:

\begin{align*}
(0) \quad |m \cdot \eta + p| > \epsilon & \quad \forall \eta \in H_k, \forall m \in \mathcal{M}_k, \forall p \in \mathbb{Z}; \\
(1) \quad q|q \eta_1 - p| > \epsilon & \quad \forall \eta \in H_k, \forall q \in Q_k^{(1)}, \forall p \in \mathbb{Z}; \\
\vdots \\
(n) \quad q|q \eta_n - p| > \epsilon & \quad \forall \eta \in H_k, \forall q \in Q_k^{(n)}, \forall p \in \mathbb{Z}
\end{align*}

for some $\epsilon > 0$.

We start by constructing a collection $(G_k^{(0)})_{k \geq 0}$ of hyperrectangles satisfying condition (0). This construction is done by induction. Then we define a subcollection $G_k^{(1)} \subset G_k^{(0)}$ of hyperrectangles that also satisfy condition (1), a subcollection $G_k^{(2)} \subset G_k^{(1)}$ that also satisfies condition (2), etc. This process ends with a subcollection $G_k^{(n)}$ that satisfies the $n$ conditions above. We would like to quantify $\#G_k^{(n)}$. We can give a lower bound, but we cannot quantify the exact cardinal. So we refine the collection $G_k^{(n)}$ by choosing a right and final subcollection $F_k$ that we can quantify.

Let

$$j_{\min} = \min_{1 \leq i \leq n} (j_i), \quad j_{\max} = \max_{1 \leq i \leq n} (j_i).$$

Let $R > 4^{1/j_{\min}}$ and $\epsilon > 0$ be such that

\begin{equation}
\epsilon < \frac{1}{2R^{2j_{\max}}}.
\end{equation}

The parameter $R$ will be chosen later to be sufficiently large in order to satisfy various conditions.

4.1. The collection $G_k^{(0)}$. For each $m \in \mathcal{M}$ and $p \in \mathbb{Z}$, let

$$\Delta(m,p) := \{ x \in \mathbb{R}^n : |m \cdot x + p| < \epsilon \}.$$

Geometrically, $\Delta(m,p)$ is the thickening of a hyperplane of the form

\begin{equation}
\mathcal{L}(m,p) := \{ x \in \mathbb{R}^n : m \cdot x + p = 0 \}
\end{equation}

with width $\epsilon/m_i$ in all the $x_i$-coordinate directions. The set $\mathcal{V}$ consists of points in $\mathbb{R}^n$ that avoid the thickening $\Delta(m,p)$ of each hyperplane $\mathcal{L}(m,p)$; alternatively, points in $\mathbb{R}^n$ that lie within any such neighbourhood are removed.
Next we describe the induction procedure in order to define the collection \((G_k^{(0)})_{k \geq 0}\). We work within the closed hypercube \(H_0 = [0,1]^n\) and set \(G_0^{(0)} = \{H_0\}\). For \(k \geq 0\), we divide each \(H_k \in G_k^{(0)}\) into new hyperrectangles \(H_{k+1}\) of size
\[
R^{-(k+1)j_1} \times \ldots \times R^{-(k+1)j_n}.
\]
Among these new hyperrectangles, we denote by \(G_k^{(0)}(H_k)\) the collection of hyperrectangles \(H_{k+1} \subset H_k\) satisfying
\[
\bullet \ G_n \subset G_{n-1}, \\
\bullet \ H_{k+1} \cap \Delta(m,p) = \emptyset \quad \forall m \in \mathcal{M}_k, \forall p \in \mathbb{Z}.
\]
We define
\[
G_{k+1}^{(0)} := \bigcup_{H_k \in G_k^{(0)}} G_k^{(0)}(H_k).
\]
Hence \(G_{k+1}^{(0)}\) is nested in \(G_k^{(0)}\) and it is a collection of ‘good’ hyperrectangles with respect to all the best approximations \(m\) satisfying \(M_m < R^k\) and all the integers \(p\). The collection \(G_k^{(0)}(H_k)\) is the collection of ‘good’ hyperrectangles that we obtain from the division of \(H_k\).

Next we give a lower bound for \(#G_k^{(0)}\). Actually, for a fixed hyperrectangle \(H_k \in G_k^{(0)}\), we give a lower bound for the number of hyperrectangles \(H_{k+1} \in G_k^{(0)}(H_k)\). Alternatively, we give an upper bound for the number of ‘bad’ hyperrectangles in \(H_k\); these are the hyperrectangles \(H_{k+1} \subset H_k\) that intersect the thickening \(\Delta(m,p)\) of some hyperplane \(L(m,p)\) with \(m \in \mathcal{M}_k\). Fact 1 and Fact 2 bound the number of thickenings \(\Delta(m,p)\) with \(m \in \mathcal{M}_k\) and \(p \in \mathbb{Z}\) that intersect \(H_k\). Fact 3 bounds the number of hyperrectangles \(H_{k+1} \subset H_k\) that each \(\Delta(m,p)\) as above intersects.

**Fact 1.** We show that for each \(k \geq 1\), the set \(\mathcal{M}_k\) contains at most \(2^{n+1}(1 + \log_2(R))\) best approximations. Let \(k \geq 0\) and \(r\) be the non-negative integer such that
\[
2^r \leq R^{k-1} < 2^{r+1}.
\]
By (18), each \(m = m_\nu \in \mathcal{M}_k\) satisfies
\[
(22) \quad M_\nu \geq 2^r.
\]
Now Lemma [2.2] implies that
\[
M_\nu + 2^n(1 + \log_2(R)) \geq 2^{1+\log_2(R)} M_\nu \\
\geq 2^{1+\log_2(R)+r} \\
> R^k.
\]
Therefore, there are at most $2^{n+1}(1 + \log_2(R))$ best approximations in $M_k$.

**Fact 2.** Fix $m \in M_k$. We show that there are at most $2^n n$ thickenings $\Delta(m, p)$ that intersect $H_k$. Indeed, suppose that two different thickenings $\Delta(m, p)$ and $\Delta(m, p')$ intersect the same edge of $H_k$. This edge of $H_k$ is a segment of a line which is parallel to an $x_l$-axis. Let $P = (y_1, \ldots, y_n)$ and $P' = (y'_1, \ldots, y'_n)$ denote the points of intersection of this line parallel to the $x_l$-axis with $L(m, p)$ and $L(m, p')$ respectively. The fact that $P$ and $P'$ both belong to a line parallel to the $x_l$-axis, implies that $y_i = y'_i \forall i \neq l$. Hence, by substracting the second equation in (23) to the first one, we have that

$$|y_l - y'_l| - \frac{2\epsilon}{|m_l|} \geq |p - p'| - \frac{2\epsilon}{|m_l|} > \frac{1}{R^{k_lj_l}} - \frac{1}{2R^{k_lj_l}} = \frac{1}{2} R^{-k_lj_l}.$$  

Since the size of $H_k$ in the $x_l$-direction is $R^{-k_lj_l}$, the inequality (24) implies that there are not more than two thickenings intersecting $H_k$. Thus the number of thickenings $\Delta(m, p)$ that intersect $H_k$ is at most twice the number of edges of $H_k$, and this is $2^n n$.

**Fact 3.** Given a thickening $\Delta(m, p)$, we give an upper bound for the number of hyperrectangles $H_{k+1} \subset H_k$ that intersect $\Delta(m, p)$. Fix $m \in M_k$ and $p \in \mathbb{Z}$. Denote by $l$ the index such that $M_m = |m_l|^{1/j_l}$. The inequalities

$$\frac{\epsilon}{|m_l|} < \frac{1}{2R^{(k+1)j_l}} \quad \text{and} \quad \frac{\epsilon}{|m_l|} \leq \frac{\epsilon}{|m_i|} \quad (1 \leq i \leq n)$$

imply that $\Delta(m, p)$ intersects at most two hyperrectangles on each line of hyperrectangles along the $x_l$-direction. Since there are $[R^{l_1}] \times \ldots \times [R^{l_i}] \times \ldots \times [R^{l_n}]$ lines of hyperrectangles along the $x_l$-direction, the thickening $\Delta(m, p)$ intersects at most

$$2[R^{l_1}] \times \ldots \times [R^{l_i}] \times \ldots \times [R^{l_n}] \leq 2R^{1-j_i}$$

hyperrectangles $H_{k+1} \subset H_k$. Hence we conclude that $\Delta(m, p)$ intersects at most $2R^{1-j_i}$ hyperrectangles $H_{k+1} \subset H_k$.

**Conclusion.** There are at most $[2^{n+1} n(1 + \log_2(R)) R^{1-j_{\min}}]$ hyperrectangles $H_{k+1} \subset H_k$ that intersect some $\Delta(m, p)$ with $m \in M_k$, where $j_{\min}$ is the minimum among the $j_i$ for all $i$.
\( p \in \mathbb{Z} \). Hence
\[
\# G^{(0)}(H_k) \geq [R] - [2^{n+1}n(1 + \log_2(R))R^{1-j_{\min}}].
\]

4.2. The subcollections \( G^{(i)} \). For each \( q \in \mathbb{N} \) and \( p \in \mathbb{Z} \), consider the sets
\[
\Gamma_i(q, p) := \{ x \in \mathbb{R}^n : q|qx_i - p| < \epsilon \} \quad (1 \leq i \leq n).
\]
Geometrically, each \( \Gamma_i(q, p) \) is a thickening of a hyperplane described by the equation \( x_i = p/q \) with width \( \epsilon/q^2 \) in the \( x_i \)-coordinate direction.

We construct a tower of subcollections
\[
G^{(n)}_k \subset \ldots \subset G^{(1)}_k \subset G^{(0)}_k,
\]
where each \( G^{(i)}_k \) consists of hyperrectangles in \( G^{(i-1)}_k \) which points avoid each thickening \( \Gamma_i(q, p) \) for \( q \in Q^{(i)}_k \). More precisely, for \( 1 \leq i \leq n \), we form \( G^{(i)}_k \) by letting
\[
G^{(i)}_k(H_k) := \left\{ H_{k+1} \in G^{(i-1)}_k(H_k) : H_{k+1} \cap \Gamma_i(q, p) = \emptyset \ \forall q \in Q^{(i)}_k \right\}
\]
and
\[
G^{(i)}_{k+1} := \bigcup_{H_k \in G^{(i-1)}_k} G^{(i)}_k(H_k).
\]
Clearly the hyperrectangles in \( G^{(i)}_{k+1} \) satisfy the conditions \((0),(1),\ldots,(i)\), so \( G^{(n)}_k \) satisfies the \( n \) conditions \((0),\ldots,(n)\).

Next, for each \( 1 \leq i \leq n \) and \( H_k \in G^{(i-1)}_k \), we give a lower bound of \( \# G^{(i)}_k(H_k) \).

Fix \( i \in \mathbb{N} \), \( i \leq n \) and \( H_k \in G^{(i-1)}_k \). Suppose that there are two pairs \((q, p)\) and \((q', p')\) in \( Q^{(i)}_k \times \mathbb{Z} \) such that
\[
H_k \cap \Gamma_i(q, p) \neq \emptyset, \quad H_k \cap \Gamma_i(q', p') \neq \emptyset.
\]
In other words, suppose there exist \( \eta, \eta' \) in \( H_k \) such that
\[
(26) \quad q|q\eta - p| < \epsilon, \quad q'|q'\eta' - p'| < \epsilon.
\]
Then, by (19) and (20), we have
\[
(27) \quad \frac{|p - p'|}{q} - \frac{\epsilon}{q^2} - \frac{\epsilon}{q'^2} > \frac{1}{qq'} - \frac{\epsilon}{q^2} - \frac{\epsilon}{q'^2} > \frac{1}{2R^{kj_i}} - \frac{1}{2R^{kj_i}} = \frac{1}{2} R^{-kj_i}.
\]
Since the size of \( H_k \) in the \( x_i \)-direction is \( R^{-kj_i} \), the inequality (27) implies that at most two thickenings of the form (25) can intersect \( H_k \).
Now, from (26), (19) and (20), it follows that if \( \eta \in \Gamma_i(q, p) \), then
\[
\left| \eta_i - \frac{p}{q} \right| < \frac{\epsilon}{q^2} < \frac{1}{2} R^{-k_j}. \]
The inequality above implies that each thickening \( \Gamma_i(q, p) \) intersects at most
\[
2[R^{j_1}] \times \ldots \times [R^{j_t}] \times \ldots \times [R^{j_n}] \leq 2R^{1-j_t}.
\]
hyperrectangles \( H_{k+1} \subset H_k \).

Therefore, there are at most \([4R^{1-j_{\min}}] \) hyperrectangles \( H_{k+1} \subset H_k \) that do not satisfy condition (i). Hence
\[
\text{(28)} \quad \#G^{(i)}(H_k) \geq [R] - [2^{n+1}n(1 + \log_2(R))R^{1-j_{\min}}] - [4iR^{1-j_{\min}}].
\]

4.3. The right subcollection \( F_k \). We choose a subcollection of \( G_k^{(n)} \) that we can exactly quantify in the following way. Let \( F_0 := G_0^{(0)} \). Choose \( R \) sufficiently large so that \([R - 2^{n+1}n(1 + \log_2(R))R^{1-j_{\min}} - 4nR^{1-j_{\min}}] > 1 \). For \( k \geq 0 \), for each \( H_k \in F_k \), we choose exactly \([R - 2^{n+1}n(1 + \log_2(R))R^{1-j_{\min}} - 4nR^{1-j_{\min}}] \) hyperrectangles from the collection \( G^{(n)}(H_k) \); denote this collection by \( F(H_k) \). Trivially,
\[
\text{(29)} \quad \#F(H_k) = [R - 2^{n+1}n(1 + \log_2(R))R^{1-j_{\min}} - 4nR^{1-j_{\min}}] > 1,
\]
so each hyperrectangle \( H_k \in F_k \) gives rise to exactly the same number of hyperrectangles \( H_{k+1} \) in \( F(H_k) \). Finally, define
\[
F_{k+1} := \bigcup_{H_k \in F_k} F(H_k).
\]
This completes the construction of the Cantor-type set
\[
K(R) := \bigcap_{k=0}^{\infty} F_k.
\]
By construction, we have \( K(R) \subset V \cap \mathbb{Bad}(1, 0, \ldots, 0) \cap \ldots \cap \mathbb{Bad}(0, \ldots, 0, 1) \). Moreover, in view of (29), we have
\[
\text{(30)} \quad \#F_{k+1} = \#F_k \#F(H_k)
\]
\[
\text{(31)} \quad = [R - 2^{n+1}n(1 + \log_2(R))R^{1-j_{\min}} - 4nR^{1-j_{\min}}]^{k+1}.
\]
5. The measure $\mu$ on $K(R)$

We now describe a probability measure $\mu$ supported on the Cantor-type set $K(R)$ constructed in the previous section. The measure we define is analogous to the probability measure used in [21] and [2] on a Cantor-type set of $\mathbb{R}^2$. For any hyperrectangle $H_k \in \mathcal{F}_k$ we attach a weight $\mu(H_k)$ which is defined recursively as follows: for $k = 0$,

$$\mu(H_0) = \frac{1}{\#\mathcal{F}_0} = 1$$

and for $k \geq 1$,

$$\mu(H_k) = \frac{1}{\#\mathcal{F}(H_{k-1})} \mu(H_{k-1}) \quad (H_k \in \mathcal{F}(H_{k-1})).$$

This procedure defines inductively a mass on any hyperrectangle used in the construction of $K(R)$. Moreover, $\mu$ can be further extended to all Borel subsets $X$ of $\mathbb{R}^n$, so that $\mu$ actually defines a measure supported on $K(R)$, by letting

$$\mu(X) = \inf \sum_{H \in \mathcal{C}} \mu(H)$$

where the infimum is taken over all coverings $\mathcal{C}$ of $X$ by rectangles $H \in \{\mathcal{F}_k : k \geq 0\}$. For further details, see [9], Proposition 1.7.

Notice that, in view of (30), we have

$$\mu(H_k) = \frac{1}{\#\mathcal{F}_k} \quad (k \geq 0).$$

A classical method for obtaining a lower bound for the Hausdorff dimension of an arbitrary set is the following mass distribution principle (see [9] p. 55).

**Lemma 5.1** (mass distribution principle). Let $\delta$ be a probability measure supported on a subset $X$ of $\mathbb{R}^n$. Suppose there are positive constants $c, s$ and $l_0$ such that

$$(32) \quad \delta(S) \leq cl^s$$

for any hypercube $S \subset \mathbb{R}^n$ with side length $l \leq l_0$. Then $\dim(X) \geq s$.

The goal in the next section is to prove that there exist constants $c$ and $l_0$ satisfying (32) with $\delta = \mu$, $X = K(R)$ and $s = n - \lambda(R)$, where $\lambda(R) \to 0$ as $R \to \infty$. Then from the mass distribution principle it will follow that $\dim(K(R)) = n$. 

6. A LOWER BOUND FOR $\dim(K(R))$

Recall that

$$j_{\text{min}} = \min_{1 \leq i \leq n} (j_i).$$

Let $k_0$ be a positive integer such that

$$R^{-kj_i} < R^{-(k+1)j_{\text{min}}} \quad \forall j_i \neq j_{\text{min}} \text{ and } k \geq k_0. \quad (33)$$

Consider an arbitrary hypercube $S$ of side length $l \leq l_0$ where $l_0$ satisfies

$$l_0 < R^{-(k_0+1)j_{\text{min}}} \quad (34)$$

together with a second inequality to be determined later. We can choose $k \geq k_0$ so that

$$R^{-(k+1)j_{\text{min}} < l \leq R^{-kj_{\text{min}}}}. \quad (35)$$

From the inequality $(33)$ it follows that

$$l > R^{-kj_i} \quad \forall j_i \neq j_{\text{min}}. \quad (36)$$

Then it is easy to see that $S$ intersects at most $2^n l^{n-1} \prod_{j_i \neq j_{\text{min}}} R^{kj_i}$ hyperrectangles $H_k \in \mathcal{F}_k$, so

$$\mu(S) \leq 2^n l^{n-1} \prod_{j_i \neq j_{\text{min}}} R^{kj_i} \mu(H_k) = 2^n l^{n-1} R^{k-j_{\text{min}}} \frac{1}{\# \mathcal{F}_k}.$$

Since $R^{(k+1)j_{\text{min}} > l^{-1}}$ (see $(35)$), we have that

$$\mu(S) \leq 2^n l^{n} R^{j_{\text{min}}} \frac{1}{\# \mathcal{F}_k}$$

$$\leq 2^n l^{n} R^{j_{\text{min}}} (1 - 4nR^{-j_{\text{min}}(2^{n-1}(1 + \log_2(R)) + 1))^{-k}}$$

by applying $(30)$. We want to choose $k$ and $\lambda(R)$ so that

$$R^{j_{\text{min}}(1 - 4nR^{-j_{\text{min}}(2^{n-1}(1 + \log_2(R)) + 1))^{-k}} \leq R^{kj_{\text{min}}\lambda(R)}. \quad (37)$$

Remember we mentioned in Section 3 that later we would choose the parameter $R$ big enough so that it satisfies various conditions. We choose $R$ so that

$$4nR^{-j_{\text{min}}(2^{n-1}(1 + \log_2(R)) + 1) \leq \frac{1}{2}. \quad \text{Then, on taking}$$

$$k \geq \log(R) \quad \text{and} \quad \lambda(R) = \frac{1 + \log(2)}{j_{\text{min}} \log(R)},$$
we have that
\[
\begin{align*}
    j_m \log(R) - k \log(1 - 4nR^{-j_m}(2^{n-1}(1+\log_2(R)) + 1)) \\
    \leq j_m \log(R) + k \log(2) \\
    \leq k(1 + \log(2)) \\
    = kj_m \lambda(R) \log(R),
\end{align*}
\]
and so the inequality (37) is satisfied. Since \( R^{kj_m} \leq l^{-1} \) (see (35)), it follows that
\[
\mu(S) \leq 2^n l^{n-\lambda(R)}.
\]
Finally, by applying the mass distribution principle we obtain
\[
\dim K(R) \geq n - \lambda(R) \to n \quad \text{as } R \to \infty.
\]

References


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