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Crossed modules of Hopf algebras and of associative algebras and two-dimensional holonomy

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Abstract

After a thorough treatment of all algebraic structures involved, we address two dimensional holonomy operators with values in crossed modules of Hopf algebras and in crossed modules of associative algebras (called here crossed modules of bare algebras.) In particular, we will consider two general formulations of the two-dimensional holonomy of a (fully primitive) Hopf 2-connection (exact and blur), the first being multiplicative the second being additive, proving that they coincide in a certain natural quotient (defining what we called the fuzzy holonomy of a fully primitive Hopf 2-connection).

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1 Introduction

A crossed module of groups $\mathcal{X} = (\partial: E \rightarrow G, \triangleright)$ is given by a map $\partial: E \rightarrow G$ of groups, together with an action \triangleright of G on E by automorphisms. This action must satisfy two natural properties (the Peiffer relations):

$$\begin{aligned} \partial(g \triangleright e) &= g \partial(e) g^{-1}, & \text{for each } g \in G, \text{ and } e \in E & & \text{(first Peiffer relation),} \\ \partial(e) \triangleright f &= e f e^{-1}, & \text{for each } e, f \in E & & \text{(second Peiffer relation).} \end{aligned}$$

Note that the second Peiffer law implies that $\ker(\partial)$ is an abelian subgroup of E . These are very flexible axioms. For example, given crossed modules $\mathcal{X} = (\partial: E \rightarrow G, \triangleright)$ and $\mathcal{X}' = (\partial': E' \rightarrow G', \triangleright')$, then $\mathcal{X} \times \mathcal{X}' = ((\partial \times \partial'): E \times E' \rightarrow G \times G', \triangleright \times \triangleright')$, with the obvious product action $\triangleright \times \triangleright'$, is a crossed module. On the other hand, if L is a subgroup of $\ker(\partial) \subset E$, such that $G \triangleright L \subset L$, which implies that L is normal (because of the second Peiffer relation), then $\mathcal{X}/L = (\partial: E/L \rightarrow G, \triangleright)$ with the obvious quotient action is also a crossed module. We note that morphisms of crossed module are defined in the obvious way.

Crossed modules of groups were invented by Whitehead in [Whi41], naturally appearing in the context of homotopy theory, being algebraic models for homotopy 2-types [MW50] (for a modern account of this see [BHS11, Bau91]). Given a pointed fibration $F \rightarrow E \rightarrow B$, then the inclusion of the fibre F in the total space E induces a crossed module $(\pi_1(F) \rightarrow \pi_1(E))$. A pointed pair of spaces (M, N) also has a fundamental crossed module $(\pi_2(M, N) \rightarrow \pi_1(N))$, with the obvious boundary map and action of π_1 on π_2 . The relation between group crossed modules and strict 2-groups (small categories whose sets of objects and of morphisms are groups, with all structure maps, including the composition, being group morphisms) was elucidated in [BS76], where we can find the first reference to the fact that the categories of crossed modules and of strict 2-groups are equivalent. This relation was fully generalized for crossed complexes and ω -groupoids in [Bro81].

We can see a (strict) 2-group as being a small 2-category with a single object where all morphisms are invertible; [BL04]. Given a crossed module $\mathcal{X} = (\partial: E \rightarrow G, \triangleright)$, the 1-morphisms of the associated 2-group,

denoted by $\mathcal{C}^\times(\mathcal{X})$, have the form $* \xrightarrow{g} *$, where $g \in G$, with the obvious composition. The 2-morphisms of $\mathcal{C}^\times(\mathcal{X})$ have the form below, composing vertically and horizontally (for conventions see 2.3.1):

$$* \begin{array}{c} \xrightarrow{\partial(e)^{-1}g} \\ \boxed{\times} \uparrow e \\ \xrightarrow{g} \end{array} * , \text{ where } g \in G, \text{ and } e \in E.$$

A general theory of (Lie) 2-groups (including non-strict ones) appears in [BL04]. This theory parallels the theory of Lie-2-algebras (strict and non-strict), which was developed in [BC04]. We mention that a strict Lie 2-algebra is uniquely represented by a crossed module $\mathfrak{X} = (\partial: \mathfrak{e} \rightarrow \mathfrak{g}, \triangleright)$ of Lie algebras (also called a differential crossed module), where \triangleright is a left action of \mathfrak{g} on \mathfrak{e} by derivations, and $\partial: \mathfrak{e} \rightarrow \mathfrak{g}$ is a Lie algebra map, furthermore satisfying the (obvious) differential Peiffer relations. Given a crossed module $\mathcal{X} = (\partial: E \rightarrow G, \triangleright)$, of Lie groups, the induced Lie algebra map $\partial: \mathfrak{e} \rightarrow \mathfrak{g}$, and action by derivations of \mathfrak{g} on \mathfrak{e} , defines a crossed module $\mathfrak{X} = (\partial: \mathfrak{e} \rightarrow \mathfrak{g}, \triangleright)$, of Lie algebras.

On the purely algebraic side, crossed modules of (Lie) groups and of Lie algebras arise in a variety of ways. Given a Lie group G , we have the actor crossed module of G , being $\mathcal{AUT}(G) = (G \xrightarrow{\text{ad}} \text{Aut}(G), \triangleright)$, where $\text{Aut}(G)$ is the Lie group of automorphisms of G , acting in G in the obvious way and ad is the map that sends every $g \in G$ to the map $\text{ad}_g: G \rightarrow G$ which is conjugation by g . If we have a central extension $\{1\} \rightarrow A \xrightarrow{i} B \xrightarrow{\partial} K \rightarrow \{1\}$ of groups, given any section $s: K \rightarrow B$ of $\partial: B \rightarrow K$, then $k \triangleright b \doteq s(k) b s(k)^{-1}$, where $k \in K$ and $b \in B$, is an action of K on B , by automorphisms independent of the chosen section, defining a crossed module with underlying group map being $\partial: B \rightarrow K$.

To a chain-complex $\mathcal{V} = (\dots \xrightarrow{\beta} V_n \xrightarrow{\beta} V_{n-1} \xrightarrow{\beta} \dots)$ of vector spaces we can associate crossed modules of Lie algebras and (in the finite dimensional case) of Lie groups, denoted respectively by $\mathfrak{GL}(\mathcal{V}) = (\partial: \mathfrak{gl}_1(\mathcal{V}) \rightarrow \mathfrak{gl}_0(\mathcal{V}), \triangleright)$ and by $\text{GL}(\mathcal{V})$; see [BL04, FMM11]. The Lie algebra $\mathfrak{gl}_0(\mathcal{V})$ is made out of chain-maps $\mathcal{V} \rightarrow \mathcal{V}$, with the usual bracket. On the other hand, the Lie algebra $\mathfrak{gl}_1(\mathcal{V})$ is given by homotopies (degree one maps) $\mathcal{V} \rightarrow \mathcal{V}$, up to 2-fold homotopies. (The bracket is not the commutator of the underlying linear maps, the latter having degree two.) The 2-group associated to the crossed module $\text{GL}(\mathcal{V})$ has a single object, has morphisms being the invertible chain maps $\mathcal{V} \rightarrow \mathcal{V}$ and 2-morphisms being the chain homotopies (up to 2-fold homotopy) between them. The vertical composition of 2-morphisms is induced by the sum of homotopies.

We can also consider pre-crossed modules of groups and of Lie algebras. These are defined similarly to crossed modules, however not imposing the second Peiffer condition. Crossed modules form a full subcategory of the category of pre-crossed modules (both in the group and Lie algebra cases). Moreover, any pre-crossed module can naturally be converted to a crossed module by dividing out the second Peiffer relations. This defines reflection functors $\{\text{pre-crossed modules}\} \rightarrow \{\text{crossed modules}\}$.

Crossed modules of groups and of Lie algebra naturally have *free* objects. Let \mathfrak{g} be a Lie algebra. Let also E be a set, with a map $\partial_0: E \rightarrow \mathfrak{g}$. We have a free differential crossed module $\mathfrak{F} = (\partial: \mathfrak{f} \rightarrow \mathfrak{g})$ on the map $\partial_0: E \rightarrow \mathfrak{g}$. Here $\partial: \mathfrak{f} \rightarrow \mathfrak{g}$ is a Lie algebra map, and we have a (set) inclusion $i: E \rightarrow \mathfrak{f}$, satisfying $\partial \circ i = \partial_0$. This free crossed module \mathfrak{F} satisfies (and is defined by) the following universal property: if we have a differential crossed module $\mathfrak{X}' = (\partial': \mathfrak{e}' \rightarrow \mathfrak{g}', \triangleright)$, a Lie algebra map $f: \mathfrak{g} \rightarrow \mathfrak{g}'$ and a set map $g_0: E \rightarrow \mathfrak{e}'$, satisfying $\partial' \circ g_0 = f \circ \partial_0$, then there exists a unique Lie algebra map $g: \mathfrak{f} \rightarrow \mathfrak{e}'$, extending g_0 , and such that, furthermore, the pair $(g: \mathfrak{f} \rightarrow \mathfrak{e}', f: \mathfrak{g} \rightarrow \mathfrak{g}')$ is a map of differential crossed modules $\mathfrak{F} \rightarrow \mathfrak{X}'$.

Models for the free differential crossed module on a set map $\partial_0: E \rightarrow \mathfrak{g}$ appear in [Ell93, CFM12b]. Consider the action \triangleright of \mathfrak{g} on $\mathcal{U}(\mathfrak{g})$, the universal enveloping algebra of \mathfrak{g} by left multiplication. Consider also the vector space $\oplus_{e \in E} \mathcal{U}(\mathfrak{g}).e$, with the obvious action \triangleright of \mathfrak{g} . Consider the linear map $\partial_1: \oplus_{e \in E} \mathcal{U}(\mathfrak{g}).e \rightarrow \mathfrak{g}$ such that $\partial_1(a.e) = a \triangleright_{\text{ad}} \partial_0(e)$ (here $a \in \mathcal{U}(\mathfrak{g})$ and $e \in E$), where $\triangleright_{\text{ad}}$ is the action of $\mathcal{U}(\mathfrak{g})$ in \mathfrak{g} induced by the adjoint action of \mathfrak{g} on \mathfrak{g} . Clearly $\partial_1(X \triangleright (a.e)) = [X, \partial_1(a.e)]$, where $X \in \mathfrak{g}$, $a \in \mathcal{U}(\mathfrak{g})$ and $e \in E$. Let \mathfrak{f}_1 be the free Lie algebra on the vector space $\oplus_{e \in E} \mathcal{U}(\mathfrak{g}).e$. We therefore have a Lie algebra map $\partial_2: \mathfrak{f}_1 \rightarrow \mathfrak{g}$, extending ∂_1 . Since any linear map $V \rightarrow V$, V a vector space, induces a derivation at the level of the free Lie algebra on V , we have a Lie algebra action \triangleright of \mathfrak{g} on \mathfrak{f}_1 , by derivations. Clearly this defines a pre-crossed module of Lie algebras. The free crossed module on the map $\partial_0: E \rightarrow \mathfrak{g}$ is obtained by converting the latter differential pre-crossed module into a crossed module of Lie algebras.

Crossed modules (of groups and of Lie algebras) are closely related to cohomology; see [WW, Bro82, Ger66]. Namely, in the group case (not considering any topology), given a group K and a K -module A , there is a one-to-one correspondence between group cohomology classes $\omega \in H^3(K, A)$ and weak homotopy

classes of crossed modules $(\partial: E \rightarrow G, \triangleright)$, fitting inside the exact sequence $\{0\} \rightarrow A \rightarrow E \xrightarrow{\partial} G \rightarrow K \cong G/\partial(E) \rightarrow \{1\}$. We actually have a group isomorphism considering Baer sums of crossed modules; [Bro82].

The same correspondence holds for Lie algebras, where we have a one-to-one correspondence between cohomology classes $\omega \in H^3(\mathfrak{k}, \mathfrak{a})$, and weak equivalence classes of crossed modules of Lie algebras $\mathfrak{X} = (\partial: \mathfrak{e} \rightarrow \mathfrak{g}, \triangleright)$, fitting inside the exact sequence $\{0\} \rightarrow \mathfrak{a} \rightarrow \mathfrak{e} \xrightarrow{\partial} \mathfrak{g} \rightarrow \mathfrak{g}/\partial(\mathfrak{e}) \cong \mathfrak{k}$; see [Wag06, Ger66, BC04].

One has to be careful that, in the Lie group case, there are several different definitions of the Lie group cohomology $H_*^3(K, A)$, for a Lie group K acting on a vector space A . For a very nice review on these topics see [WW, BL04]. A version of Lie-group cohomology $H_g(K, A)$ whose obvious linearization yields an isomorphism $H_g^3(K, A) \cong H^3(\mathfrak{k}, \mathfrak{a})$ is obtained by considering germs of group cocycles $\omega: K^n \rightarrow A$, where a cocycle germ satisfies the same laws of the usual group cohomology, except that it is only defined in a neighborhood of the identity (in addition to being smooth), and two of these considered to be equivalent if they coincide in a neighborhood of the identity. There is a map $H_g^n(K, A) \rightarrow H^n(\mathfrak{k}, \mathfrak{a})$ induced by derivation which is an isomorphism. If K is compact, these local group cocycles never extend to everywhere defined, cohomologically non-trivial, continuous cocycles. The reason is that if $\omega: K^{n+1} \rightarrow K$ is a continuous group cocycle then $(a_1, \dots, a_n) \in G^n \xrightarrow{\omega} \int_G \omega(a_1, \dots, a_n, g) dg$ is such that $d\alpha = \pm\omega$. Nevertheless, for integral cohomology classes, one can find extensions to globally defined cocycles, however not continuous outside of a neighborhood of the identity. These integral cohomology classes ω can frequently be realized by crossed modules $(P \rightarrow B, \triangleright)$, of (infinite dimensional) Lie groups, fitting inside the exact sequence: $\{a\} \rightarrow A \rightarrow P \rightarrow B \rightarrow K \rightarrow \{1\}$.

Consider a compact semisimple Lie group G with Lie algebra \mathfrak{g} . We have a cohomology class $\omega: \mathfrak{g}^{\otimes 3} \rightarrow \mathbb{C}$, given by $\omega(X, Y, Z) = \langle [X, Y], Z \rangle$, where $\langle \cdot, \cdot \rangle$ is the Cartan-Killing form. A geometric realization of the class $v = 2\pi i \omega$ as being the differential crossed module associated to a crossed module of infinite dimensional Lie groups, denoted by String, appears in [BSCS07]. In the $SU(2) \cong S^3$ case, this crossed module of Lie groups fits inside the exact sequence $\{0\} \rightarrow U(1) \xrightarrow{i} \widehat{\Omega(S^3)} \xrightarrow{\partial} \mathcal{P}(S^3) \xrightarrow{p} SU(2)$. Here $\mathcal{P}(S^3)$ denotes the group of smooth paths $[0, 1] \rightarrow S^3$, starting at the identity, with point-wise product, p being the map choosing the final point of the path; $\widehat{\Omega(S^3)}$ is the Kac-Moody group, defining (see [Mur88, PS86]) a universal central extension $U(1) \rightarrow \widehat{\Omega(S^3)} \xrightarrow{p} \Omega(S^3)$ of the loop group $\Omega(S^3)$, the group of smooth paths $[0, 1] \rightarrow S^3$, starting and ending at the identity. The boundary map $\partial: \widehat{\Omega(S^3)} \rightarrow \mathcal{P}(S^3)$ is the composition of the projection map $p: \widehat{\Omega(S^3)} \rightarrow \Omega(S^3)$ and the inclusion map $\Omega(S^3) \rightarrow \mathcal{P}(S^3)$.

A general theory of 2-bundles was initiated in [BS05], with very strong emphasis on their two dimensional holonomy. These 2-bundles have a structure Lie 2-group (equivalent to a Lie crossed module), which (as far as two dimensional holonomy is concerned) is taken to be strict, however not necessarily finite dimensional. We note that 2-bundles with connection are closely related with gerbes with connection; see [BM05]. Further work on the holonomy of a 2-bundle with a 2-connection appears in [SW11, SW13, FP10, FP11a, FP11b].

Let $\mathcal{X} = (\partial: E \rightarrow G, \triangleright)$ be a crossed module of Lie groups, with associated differential crossed module $\mathfrak{X} = (\partial: \mathfrak{e} \rightarrow \mathfrak{g}, \triangleright)$. Given a 2-bundle P over a manifold M , with structure 2-group \mathcal{X} , a 2-connection in P is locally given by a 1-form ω in M with values in \mathfrak{g} , and an \mathfrak{e} -valued 2-form m , in M , such that $\partial(m) = \mathcal{F}_\omega$, the curvature of ω , in our conventions: $\mathcal{F}_\omega = d\omega - \frac{1}{2}\omega \wedge \omega$. The 2-curvature 3-form of the pair (ω, m) is, by definition, $\mathcal{M} = dm - \omega \wedge m$. A 2-connection is said to be flat if its 2-curvature 3-form vanishes.

Let us roughly explain what we mean by the two dimensional holonomy of a local 2-connection (ω, m) , referring details to the text and the references mentioned above. Let M be a manifold. We denote by $\mathcal{P}^1(M)$ the space of continuous and piecewise smooth paths $\gamma: [0, 1] \rightarrow M$ (1-paths), represented as:

$$x = \gamma(0) \xrightarrow{\gamma} \gamma(1) = y.$$

If γ_1 and γ_2 are 1-paths with $\gamma_1(1) = \gamma_2(0)$, we define their concatenation, written as $(x \xrightarrow{\gamma_1} y \xrightarrow{\gamma_2} z) = x \xrightarrow{\gamma_1 \gamma_2} z$, in the obvious way. We define a 2-path as being a map $\Gamma: [0, 1]^2 \rightarrow M$, which is to be continuous and piecewise smooth, for some paving of the square by polygons, transverse to the boundary of the square. We impose, moreover, that $\Gamma(\{0, 1\} \times [0, 1])$ has at most two elements. Putting $\{x\} = \Gamma(\{0\} \times [0, 1])$, $\{y\} = \Gamma(\{1\} \times [0, 1])$, and also $\gamma_1(t) = \Gamma(t, 1)$, $\gamma_0(t) = \Gamma(t, 0)$, where $t \in [0, 1]$, defining 1-paths $\gamma_0, \gamma_1: [0, 1] \rightarrow M$,

we denote the 2-path Γ in the (very suggestive) form:

$$\Gamma = x \begin{array}{c} \xrightarrow{\gamma_1} \\ \uparrow \Gamma \\ \xrightarrow{\gamma_0} \end{array} y .$$

The set of 2-paths in M is denoted by $\mathcal{P}^2(M)$. Clearly 2-paths compose horizontally and vertically, having obvious horizontal and vertical reverses. We also have whiskerings of 2-paths by 1-paths.

Let $\mathcal{P}(M)$ be the triple $(\mathcal{P}^2(M), \mathcal{P}^1(M), M)$, with the natural compositions, boundaries and reverses. If $\mathcal{X} = (\partial: E \rightarrow G, \triangleright)$ is a crossed module of Lie groups, with tangent differential crossed module $\mathfrak{X} = (\partial: \mathfrak{e} \rightarrow \mathfrak{g}, \triangleright)$, then a pair of differential forms $(\omega \in \Omega^1(M, \mathfrak{g}), m \in \Omega^2(M, \mathfrak{e}))$, in M , with $\partial(m) = \mathcal{F}_\omega$, defines a local holonomy operator $\mathcal{F}_{(\omega, m)}^\times$. This operator assign to every 2-path Γ a 2-morphism of the 2-group $\mathcal{C}^\times(\mathcal{X})$ associated with \mathcal{X} . Denote it by:

$$\mathcal{F}_{(\omega, m)}^\times \left(x \begin{array}{c} \xrightarrow{\gamma_2} \\ \uparrow \Gamma \\ \xrightarrow{\gamma_1} \end{array} y \right) = * \begin{array}{c} \xrightarrow{(P_\omega(\gamma_2))^{-1}} \\ \boxed{\times} \uparrow (Q_{(\omega, m)}(\Gamma))^{-1} \\ \xrightarrow{(P_\omega(\gamma_1))^{-1}} \end{array} * . \quad (1)$$

The operator $\mathcal{F}_{(\omega, m)}^\times$ preserves all compositions, boundaries and reverses, in $\mathcal{P}(M)$ and in $\mathcal{C}^\times(\mathcal{X})$.

In the context of graded vector bundles, a different type of higher dimensional holonomy (of representations up to homotopy) was developed in [AC12, AS14b]. In a natural (non-trivial) quotient (cotruncation in dimension two) the two-dimensional part of the holonomy of a representation up to homotopy coincides with the two-dimensional holonomy of a naturally associated 2-connection, as recently proved in [AS14a]. A similar result is mentioned in [CFM12a]. Later on, will also give a new insight to this non-trivial fact.

In this paper we propose a fully algebraic construction of the two dimensional holonomy operators. The main reason for doing this is the following: (local) 2-connections in a smooth manifold M may arise simply as a pair $(\omega \in \Omega^1(M, \mathfrak{g}), m \in \Omega^2(M, \mathfrak{e}))$, with $\partial(m) = \mathcal{F}_\omega$, the curvature of ω , where $\mathfrak{X} = (\partial: \mathfrak{e} \rightarrow \mathfrak{g}, \triangleright)$ is a differential crossed module, with no obvious crossed module of Lie groups associated to it. The Lie algebras \mathfrak{g} and \mathfrak{e} may very well be infinite dimensional, and without any assigned or natural topology, which makes it impossible to apply Lie third theorem, in order to find a Lie crossed module \mathcal{X} integrating \mathfrak{X} .

Examples of such infinite dimensional Lie algebras and differential crossed module arise as quotients of free Lie algebras and of free differential crossed modules. An important case is the Lie algebra \mathfrak{ch}_n (here n is a positive integer), the Lie algebra of horizontal chord diagrams in n strands, which is formally generated by the symbols r_{ab} , where $1 \leq a, b \leq n$, satisfying the infinitesimal braid group relations; [Kas95, Bar95]:

$$\begin{aligned} r_{ab} &= r_{ba}, \\ [r_{ab}, r_{cd}] &= 0 \text{ for } \{a, b\} \cap \{c, d\} = \emptyset, \\ [r_{ab} + r_{ac}, r_{bc}] &= 0 = [r_{ab}, r_{ac} + r_{bc}] \quad (\text{called the "four term relations"}). \end{aligned}$$

The configuration space $\mathbb{C}(n)$ of n (distinguishable) particles in the complex plane is, by definition, $\mathbb{C}(n) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_i \neq z_j \text{ if } i \neq j\}$. Consider the following \mathfrak{ch}_n -valued connection form in $\mathbb{C}(n)$, called the universal Knizhnik-Zamolodchikov (KZ) connection, which is a flat connection, $\mathcal{F}_A = 0$:

$$A = \sum_{1 \leq a < b \leq n} \omega_{ab} r_{ab}, \quad \text{where } \omega_{ab} = \frac{dz_a - dz_b}{z_a - z_b}. \quad (2)$$

(We mention that the Kontsevich Integral [Kon93, Bar95] is essentially given by the holonomy of the connection form A .) Since there is no Lie group naturally assigned to \mathfrak{ch}_n , we must adress the holonomy of A in an algebraic fashion. We can do this is by introducing a formal parameter h , passing to the connection form $A_h = \sum_{1 \leq a < b \leq n} h \omega_{ab} r_{ab}$. This connection is flat in a strong (graded) sense: $dA_h + \frac{1}{2}(A_h \wedge A_h) = 0$, since $dA = 0$ and $A \wedge A = 0$. Then given a path $\gamma: [0, 1] \rightarrow \mathbb{C}(n)$, the holonomy of A_h , over γ , it is given by the solution of the differential equation $\frac{d}{dt} a(t) = h A(\gamma'(t)) a(t)$, $a(0) = 1$, in the space $\mathcal{U}(\mathfrak{ch}_n)[[h]]$ of formal power series over the universal enveloping algebra $\mathcal{U}(\mathfrak{ch}_n)$ of \mathfrak{ch}_n . This has a unique solution, which can be expressed by using Chen's iterated integrals; [Che73] (we fully review the construction here). The

map $t \mapsto a(t)$ takes values in the group of group like elements of $\mathcal{U}(\mathfrak{ch}_n)[[h]]$, namely $\Delta(a(t)) = a(t) \otimes a(t)$, where Δ is the coproduct map in the usual Hopf algebra structure given to $\mathcal{U}(\mathfrak{ch}_n)$, extended to $\mathcal{U}(\mathfrak{ch}_n)[[h]]$.

Categorifications, in the realm of differential crossed modules, of the Lie algebra of horizontal chord diagrams were addressed in [CFM12b, CFM12a]. The calculations in [CF15] indicate that the differential crossed module $2\mathfrak{ch}_n = (\partial: 2\mathfrak{ch}_n \rightarrow \mathfrak{ch}_n^+, \triangleright)$ associated to the Lie 2-algebra of horizontal categorified chord diagrams in n strands should be defined in the following way. Firstly, \mathfrak{ch}_n^+ is the quotient of the free Lie algebra on the symbols r_{ab} ($1 \leq a < b \leq n$) by the relations:

$$r_{ab} = r_{ba}, \quad [r_{ab}, r_{cd}] = 0 \text{ for } \{a, b\} \cap \{c, d\} = \emptyset.$$

(Note that the 4-term relations are not imposed.) In order to construct $2\mathfrak{ch}_n$, consider the free differential crossed module $(\partial: 2\mathfrak{ch}_n^+ \rightarrow \mathfrak{ch}_n^+, \triangleright)$ on the symbols P_{abc} and Q_{abc} , with $1 \leq a < b < c \leq n$, such that:

$$\partial(P_{abc}) = [r_{bc}, r_{ab} + r_{ac}], \quad \partial(Q_{abc}) = [r_{ab}, r_{ac} + r_{bc}].$$

And then we form the quotient $2\mathfrak{ch}_n$ of $2\mathfrak{ch}_n^+$ by the \mathfrak{ch}_n^+ module generated by the relations below:

$$\begin{aligned} r_{ab} \triangleright P_{ijk} &= 0 = r_{ab} \triangleright Q_{ijk}, \text{ if } \{a, b\} \cap \{i, j, k\} = \emptyset \\ (r_{ad} + r_{bd} + r_{cd}) \triangleright P_{abc} - (r_{ab} + r_{ac}) \triangleright Q_{bcd} + r_{bc} \triangleright (Q_{abd} + Q_{acd}) &= 0, \\ (r_{ab} + r_{ac} + r_{ad}) \triangleright P_{bcd} + r_{cd} \triangleright (P_{abc} + P_{abd}) - (r_{bc} + r_{bd}) \triangleright P_{acd} &= 0, \\ (r_{ad} + r_{bd} + r_{cd}) \triangleright Q_{abc} + r_{ab} \triangleright (Q_{acd} + Q_{bcd}) - (r_{ac} + r_{bc}) \triangleright Q_{abd} &= 0, \\ (r_{ab} + r_{ac} + r_{ad}) \triangleright Q_{bcd} + r_{bc} \triangleright (P_{abd} + P_{acd}) - (r_{bd} + r_{cd}) \triangleright P_{abc} &= 0, \\ r_{ab} \triangleright (P_{acd} + P_{bcd}) - r_{cd} \triangleright (Q_{abc} + Q_{abd}) &= 0, \\ r_{ac} \triangleright (P_{abd} - P_{bcd} - Q_{bcd}) + r_{bd} \triangleright (Q_{abc} + P_{abc} - Q_{acd}) &= 0. \end{aligned}$$

Note that the left-hand-side of each of these relations maps to zero in \mathfrak{ch}_n^+ , so that the quotient is well defined (for a proof see [CFM12b].) Therefore we have a crossed module $2\mathfrak{ch}_n = (\partial: 2\mathfrak{ch}_n \rightarrow \mathfrak{ch}_n^+, \triangleright)$.

It is proven in [CFM12b, CF15] that the pair (A, B) , where A is as in (2), and B is as below, is a flat 2-connection in the configuration space of n particles in \mathbb{C} , called the universal KZ 2-connection:

$$B = 2 \sum_{1 \leq a < b < c \leq n} \omega_{bc} \wedge \omega_{ca} P_{abc} - 2 \sum_{1 \leq a < b < c \leq n} \omega_{ca} \wedge \omega_{ab} Q_{abc}.$$

Our previous work on the categorification of the Lie algebra of horizontal chord diagrams, and the strongly related construction of the KZ-2-connection, was motivated by the possible applications to the categorification of the Kontsevich Integral, in order to extend it, at least partially, to invariants of knotted surfaces in the 4-sphere. The project was also motivated by the issue of constructing categorified Drinfeld associators, presumably important for constructing explicit examples of braided monoidal bicategories. (In the uncategorified setting, Drinfeld associators can also be constructed from the holonomy of the KZ-connection). Both of these applications require the definition of the two dimensional holonomy of the KZ-2-connection. Given that there is no crossed module of Lie groups associated to the differential crossed module $2\mathfrak{ch}_n$ of categorified horizontal chord diagrams a fully algebraic definition of this surface holonomy is very much needed.

1.1 Plan of the paper

The algebraic and categorical foundations of this paper will be quite heavy, occupying the biggest part of it. We note that the word ‘‘count’’ will be used both for designating categorical counts (in the context of adjoint functors) and for designating coalgebraic counts (in the context of Hopf algebras.) Compositions in the text are intended from right to left.

1.1.1 Algebraic setting: one dimensional holonomy

We will consider three types of associative algebras: unital algebras, where morphisms of unital algebras will, by imposition, always preserve identities, Hopf algebras, and bare algebras, which are associative algebras,

not considering any other structure that might exist there (identity, coproduct, counit, etc). There is a pair of adjoint functors, where U is the obvious underlying bare algebra (forgetful) functor:

$$\{\text{bare algebras}\} \begin{array}{c} \xrightarrow{(\cdot)_\bullet} \\ \xrightarrow{\xi_\bullet} \\ \perp \\ \xleftarrow{U} \end{array} \{\text{unital algebras}\} . \quad (3)$$

(Identity functors will not appear in the diagrams we draw. However they are always implicit, normally being targets of natural transformations, as above. Natural transformations will be denoted by double arrows $\Longrightarrow \cdot$.) The left adoint $(\cdot)_\bullet$ of the adjunction (3) adjoins an identity to a bare algebra A , yielding $A_\bullet = \mathbb{C} \times A$, with product $(\lambda, a)(\lambda', a) = (\lambda\lambda', \lambda a' + \lambda' a + aa')$, where $a, a' \in A$ and $\lambda, \lambda' \in \mathbb{C}$. The counit $\xi_\bullet: (\cdot)_\bullet \circ U \Longrightarrow \text{id}$, of the adjunction, associates to a unital algebra A the map $\xi_{\bullet A}: A_\bullet \rightarrow A$, such that $\xi_{\bullet A}(\lambda, a) = \lambda 1_A + a$, where $a \in A$ and $\lambda \in \mathbb{C}$. (Here 1_A is the identity of A .)

We recall the well known adjunction between unital algebras and Lie algebras:

$$\{\text{Lie algebras}\} \begin{array}{c} \xrightarrow{\mathcal{U}} \\ \xrightarrow{\xi} \\ \perp \\ \xleftarrow{\text{Lie}} \end{array} \{\text{unital algebras}\} . \quad (4)$$

Here \mathcal{U} is the universal enveloping algebra functor, and Lie associates to any unital algebra (or, more generally, bare algebra) A its commutator Lie algebra: $[a, b] = ab - ba$, for each $a, b \in A$. Denote by $\xi: \mathcal{U} \circ \text{Lie} \Longrightarrow \text{id}$ the counit of this adjunction, defining, for each unital algebra A , an algebra map $\xi_A: \mathcal{U}(\text{Lie}(A)) \rightarrow A$. Recall that $\mathcal{U}(\mathfrak{g})$, where \mathfrak{g} is a Lie algebra, is a quotient of the tensor algebra $\mathcal{T}(\mathfrak{g}) = \mathbb{C} \oplus \bigoplus_{n=1}^{+\infty} \mathfrak{g}^{\otimes n}$ by the biideal generated by $\{XY - YX - [X, Y], X, Y \in \mathfrak{g}\}$. It is well known that $\mathcal{U}(\mathfrak{g})$ is a Hopf algebra, where every element of $\mathfrak{g} \subset \mathcal{U}(\mathfrak{g})$ is primitive: $\Delta(X) = X \otimes 1 + 1 \otimes X$, for each $X \in \mathfrak{g}$. The (Hopf algebra) counit satisfies $\epsilon(X) = 0, \forall X \in \mathfrak{g}$.

The adjunction (4), between unital algebras and Lie algebras, factors through another adjunction, between the categories of bare algebras and of Lie algebras, namely:

$$\{\text{Lie algebras}\} \begin{array}{c} \xrightarrow{\mathcal{U}^0} \\ \xrightarrow{\xi^0} \\ \perp \\ \xleftarrow{\text{Lie}} \end{array} \{\text{bare algebras}\} . \quad (5)$$

Here $\mathcal{U}^0(\mathfrak{g}) = \ker(\epsilon: \mathcal{U}(\mathfrak{g}) \rightarrow \mathbb{C})$ is the augmentation ideal of $\mathcal{U}(\mathfrak{g})$. Therefore $\mathcal{U}^0(\mathfrak{g})$ does not have a unit, but, consequence of the Poincaré-Birkhoff-Witt Theorem, we obtain $\mathcal{U}(\mathfrak{g})$ by formally adjoining a unit to $\mathcal{U}^0(\mathfrak{g})$. Given a bare algebra A , we have an obvious map (the flat counit) $\xi_A^0: \mathcal{U}^0(\text{Lie}(A)) \rightarrow A$. The top and the bottom of the following diagram commute, and this goes over to (adjunction) units and counits:

$$\begin{array}{ccccc} & & \mathcal{U} & & \\ & \curvearrowright & & \curvearrowleft & \\ \{\text{Lie algebras}\} & \xrightarrow{\mathcal{U}^0} & \{\text{bare algebras}\} & \xrightarrow{(\cdot)_\bullet} & \{\text{unital algebras}\} \\ & \perp & & \perp & \\ & \xleftarrow{\text{Lie}} & & \xleftarrow{U} & \\ & & \perp & & \\ & & \text{Lie} & & \end{array} \quad (6)$$

If A is a bare algebra (without unit) there cannot exist a surjective algebra map $\mathcal{U}(\text{Lie}(A)) \rightarrow A$. However, clearly, there is a natural transformation ξ_\sharp (the sharp counit) between the functors $\mathcal{U} \circ \text{Lie}$ and $(\cdot)_\bullet$:

$$\begin{array}{ccccc} & & \mathcal{U} & & \\ & \curvearrowright & & \curvearrowleft & \\ \{\text{Lie algebras}\} & \xrightarrow{\mathcal{U}^0} & \{\text{bare algebras}\} & \xrightarrow{(\cdot)_\bullet} & \{\text{unital algebras}\} \\ & \perp & & \perp & \\ & \xleftarrow{\text{Lie}} & & \xleftarrow{U} & \\ & & \xi_\sharp & & \end{array} \quad (7)$$

For a bare algebra A , $(\xi_{\#})_A: \mathcal{U}(\text{Lie}(A)) \rightarrow A_{\bullet}$ on generators sends $a \in A$ to $(0, a)$ and $1_{\mathcal{U}(\text{Lie}(A))}$ to $(1, 0)$.

There is a functor Prim from the category of Hopf algebras to the category of Lie algebras, sending a Hopf algebra H to its Lie algebra $\text{Prim}(H)$ of primitive elements. We have a functor $(\cdot)_{\mathfrak{gl}}^*$ from the category of Hopf algebras to the category of groups, associating to each Hopf algebra H the group $H_{\mathfrak{gl}}^* \subset H[[\hbar]]$ of group like invertible elements of $H[[\hbar]]$ (the algebra of formal power series over H), such that the first term of the power series is 1_H , the identity of H . (The holonomy of a \mathfrak{g} valued connection form in M essentially takes values in $\mathcal{U}(\mathfrak{g})_{\mathfrak{gl}}^*$.) There is another functor $(\cdot)^*$, from the category of unital algebras to the category of groups, associating to a unital algebra A the group A^* of formal power series in $A[[\hbar]]$, starting in 1_A .

In the uncategorified setting, the diagram of all categories, functors and natural transformations which we will use is below; (8). Note that U always denotes the obvious forgetful functor.

$$\begin{array}{c}
 \text{Lie} \\
 \top \\
 \mathcal{U} \\
 \begin{array}{ccc}
 \{\text{Lie algebras}\} & \xrightarrow{\xi^b} & \{\text{bare algebras}\} & \xrightarrow{\xi_{\bullet}} & \{\text{unital algebras}\} \\
 \downarrow \text{Lie} & \perp & \downarrow U & \perp & \downarrow U \\
 \{\text{Hopf Algebras}\} & & & & \{\text{Groups}\} \\
 \uparrow \text{Prim} & & & & \downarrow (\cdot)_{\mathfrak{gl}}^* \\
 & & & & \{\text{Groups}\}
 \end{array}
 \end{array}
 \quad (8)$$

In (8), sub-diagrams where all arrows are left pointing commute. The same holds for right-pointing diagrams, except for the triangle in the bottom right. We have a natural transformation $\iota: (\cdot)^* \circ U \implies (\cdot)_{\mathfrak{gl}}^*$, filling the triangle below. As before, ξ is the counit of the adjunction (4) between Lie algebras and unital algebras.

$$\begin{array}{c}
 \text{Lie} \\
 \top \\
 \xi \\
 \begin{array}{ccc}
 \{\text{Lie algebras}\} & \xrightarrow{\xi} & \{\text{unital algebras}\} \\
 \downarrow \mathcal{U} & \nearrow U & \downarrow U \\
 \{\text{Hopf Algebras}\} & & \{\text{Groups}\} \\
 & & \downarrow (\cdot)_{\mathfrak{gl}}^* \\
 & & \{\text{Groups}\}
 \end{array}
 \end{array}
 \quad (9)$$

Here, if H is an Hopf algebra, then ι_H is the group inclusion $H_{\mathfrak{gl}}^* \rightarrow H^*$. Composing the natural transformations ξ and ι in (9), we have a natural transformation of functors $\{\text{unital algebras}\} \rightarrow \{\text{groups}\}$:

$$\text{proj}: (\cdot)_{\mathfrak{gl}}^* \circ U \circ \text{Lie} \implies (\cdot)^*.$$

Explicitly, if A is a unital algebra, one has the counit $\xi_A: \mathcal{U}(\text{Lie}(A)) \rightarrow A$, extending to an algebra map $\xi_A: \mathcal{U}(\text{Lie}(A))[[\hbar]] \rightarrow A[[\hbar]]$. This clearly sends $\mathcal{U}(\text{Lie}(A))_{\mathfrak{gl}}^*$ to A^* .

If we have a connection form $\omega \in \Omega^1(M, A)$, where M is a manifold and A a unital algebra, then we can address the holonomy of ω , either directly over $A[[\hbar]]$, taking values in A^* , or over $\mathcal{U}(\text{Lie}(A))[[\hbar]]$, taking values in $\mathcal{U}(\text{Lie}(A))_{\mathfrak{gl}}^*$. In both cases, given a path $\gamma: [0, 1] \rightarrow M$ we solve the differential equation $\frac{d}{dt} P_{\omega}(\gamma, [t, t_0]) = \hbar \omega(\gamma'(t)) P_{\omega}(\gamma, [t, t_0])$, with $P_{\omega}(\gamma, [t_0, t_0]) = 1$, however firstly over $\mathcal{U}(\text{Lie}(A))[[\hbar]]$, and in the latter case over $A[[\hbar]]$. In the first case the holonomy takes values in $\mathcal{U}(\text{Lie}(A))_{\mathfrak{gl}}^*$, whereas in the second case it takes values over A^* . These two holonomies are related by the map $\text{proj}_A: \mathcal{U}(\text{Lie}(A))_{\mathfrak{gl}}^* \rightarrow A^*$.

1.1.2 Algebraic and categorical setting: two-dimensional holonomy

In order to consider two dimensional holonomies, we first extend the previous diagrams (8) of categories, functors and natural transformations to the categorified (crossed module) setting. We will consider in total five categories of crossed modules, namely: crossed modules of Lie algebras and of (Lie) groups (already mentioned), and, in addition, crossed modules of Hopf algebras, crossed modules of bare algebras and unital crossed modules of bare algebras. The pattern of the construction is always the same. For example, a crossed module $\mathcal{H} = (\partial: I \rightarrow H, \rho)$, of Hopf algebras [Maj12], is given by a Hopf algebra map $\partial: I \rightarrow H$, and a linear map $\rho: H \otimes I \rightarrow I$, making I into an H -module algebra and coalgebra (and ρ should satisfy the obvious Hopf analogues of the first and second Peiffer relations.)

A crossed module $\mathcal{A} = (\partial: A \rightarrow B, \triangleright, \triangleleft)$ of bare algebras is given by a bare algebra map $\partial: A \rightarrow B$, together with two actions \triangleright and \triangleleft , on the left and on the right, of B on A , by (left or right) multipliers. The left and right actions are to be compatible, each furthermore satisfying the first and second Peiffer relations. A crossed module of bare algebras is said to be unital if the bottom algebra B is unital, with an identity 1_B satisfying $1_B \triangleright a = a = a \triangleleft 1_B$, for each $a \in A$. If (possibly) only the first Peiffer relation is satisfied by \triangleright and \triangleleft then $\hat{\mathcal{A}} = (\partial: A \rightarrow B, \triangleright, \triangleleft)$ will be called a pre-crossed module of bare algebras, and similarly for Hopf algebras. As in the case of groups and Lie algebras, a pre-crossed module $\hat{\mathcal{A}} = (\partial: A \rightarrow B, \triangleright, \triangleleft)$, of bare algebras, can be converted (by a reflection functor \mathcal{R}) into a crossed module of bare algebras $\mathcal{R}(\hat{\mathcal{A}}) = (\partial: \underline{A} \rightarrow B, \triangleright, \triangleleft)$. We obtain \underline{A} from A by dividing out the ideal generated by the second Peiffer identities.

We have the following diagram of categories and functors, where the acronyms “C.M.” and “P.C.M.” mean crossed modules and pre-crossed modules, extending some of the instances of the previous discussion to the “crossed module” context, with some new categories and functors added; c.f. (8). Full explanation is below.

$$\begin{array}{ccccc}
 & & \text{Lie} & & \\
 & & \top & & \\
 & \swarrow & & \searrow & \\
 \{\text{Lie algebras C.M.}\} & \xleftarrow{\text{Lie}} & \{\text{bare algebras C.M.}\} & \xleftarrow{U} & \{\text{unital bare algebras C.M.}\} \\
 \begin{array}{c} \left(\begin{array}{c} \downarrow \\ \uparrow \\ \downarrow \end{array} \right) \\ \text{u} \quad \text{Prim} \end{array} & & & & \\
 \{\text{Hopf Algebras C.M.}\} & \xrightarrow{\widehat{\mathcal{B}\mathcal{A}}} & \{\text{unital bare algebras P.C.M.}\} & \xrightarrow{\mathcal{R}} & \{\text{unital bare algebras C.M.}\} \\
 & \searrow \text{()}_{\text{gl}}^* & & \searrow \text{()}_* & \\
 & & \{\text{Groups C.M.}\} & \xrightarrow{c^\times()} & \{\text{2-groupoids}\} \\
 & & & & \downarrow c^+() \\
 & & & & \{\text{2-groupoids}\}
 \end{array} \tag{10}$$

(Left pointing arrows commute. Right pointing arrows commute.) Let us briefly explain the diagram (leaving full details for the paper itself). We have a forgetful functor U from the category of unital crossed module of bare algebras to the category of crossed modules of bare algebras. A crossed module of bare algebras $\mathcal{A} = (\partial: A \rightarrow B, \triangleright, \triangleleft)$ gives rise to a crossed module of Lie algebras, simply as: $\text{Lie}(\mathcal{A}) = (\partial: \text{Lie}(A) \rightarrow \text{Lie}(B), \triangleright)$, where $b \triangleright a = b \triangleright a - a \triangleleft b$, for $b \in B$ and $a \in A$. Suppose that \mathcal{A} is unital, thus that B is a unital algebra. Let $A_\bullet^* \subseteq \text{Units}(A_\bullet[[h]])$, where “Units” denotes the group of invertible elements, be the group of formal power series in $A_\bullet = \mathbb{C} \times A$, whose first term is $1_{\mathbb{C}} = (1_{\mathbb{C}}, 0)$, and such that all other terms are h^n ($n \geq 1$) multiples of elements of $A = \{0\} \times A$. We have a crossed module of groups: $(\mathcal{A})_\bullet^* = (\partial: A_\bullet^* \rightarrow B^*, \triangleright)$, where, as before, $B^* \subset \text{Units}(B[[h]])$ is the group of formal power series starting in 1_B .

A crossed module of Lie algebras $\mathfrak{X} = (\partial: \mathfrak{e} \rightarrow \mathfrak{g}, \triangleright)$ gives rise to a crossed module of Hopf algebras $\mathcal{U}(\mathfrak{X}) = (\partial: \mathcal{U}(\mathfrak{e}) \rightarrow \mathcal{U}(\mathfrak{g}), \rho)$, [Maj12], in a natural way. Passing to the Lie algebras of primitive elements, if $\mathcal{H} = (\partial: I \rightarrow H, \rho)$ is a Hopf algebra crossed module, then $\text{Prim}(\mathcal{H}) = (\partial: \text{Prim}(I) \rightarrow \text{Prim}(H), \triangleright)$ is a differential crossed module. Considering the groups of group-like elements in $H[[h]]$ and $I[[h]]$, whose first term of the power series is the identity, yields a crossed module of groups $(\mathcal{H})_{\text{gl}}^* = (\partial: I_{\text{gl}}^* \rightarrow H_{\text{gl}}^*, \triangleright)$.

There is also a functor $\widehat{\mathcal{B}\mathcal{A}}$ sending a crossed module $\mathcal{H} = (\partial: I \rightarrow H, \rho)$, of Hopf algebras, to a pre-crossed module of bare algebras. Explicitly: $\widehat{\mathcal{B}\mathcal{A}}((\partial: I \rightarrow H, \rho)) = (t: I^0 \otimes_\rho H \rightarrow H, \triangleright, \triangleleft)$, where I^0 is the augmentation ideal (the kernel of the counit) of I , and $I^0 \otimes_\rho H$ is the smash product as an algebra and a product coalgebra. Also $t(a \otimes x) = \partial(a)x$, where $a \in I$ and $x \in H$. The natural inclusion $H \rightarrow I \otimes_\rho H$ turns $I^0 \otimes_\rho H$ into a left and a right module over H . Composing $\widehat{\mathcal{B}\mathcal{A}}$ with \mathcal{R} gives a functor $\mathcal{B}\mathcal{A}$, associating a crossed module of bare algebras to a crossed module of Hopf algebras. We have an adjunction between the categories of unital crossed modules of bare algebras and of crossed modules of Lie algebras, where

$\mathcal{BA} \circ \mathcal{U} = \mathcal{R} \circ \widehat{\mathcal{BA}} \circ \mathcal{U}$ is the left adjoint:

$$\begin{array}{ccc}
 & \text{Lie} & \\
 & \top & \\
 \{\text{Lie algebras C.M.}\} & \xleftarrow{\quad} & \{\text{unital bare algebras C.M.}\} \\
 \mathcal{U} \left(\begin{array}{c} \uparrow \\ \{\text{Hopf Algebras C.M.}\} \end{array} \right. & \xrightarrow{\quad \mathcal{BA} \quad} & \{\text{unital bare algebras P.C.M.}\} \\
 & \xrightarrow{\quad \mathcal{BA} \quad} & \xrightarrow{\quad \mathcal{R} \quad}
 \end{array}
 \tag{11}$$

Given a unital crossed module of bare algebras $\mathcal{A} = (\partial: A \rightarrow B, \triangleright, \triangleleft)$, we have a map of pre-crossed modules of bare algebras: $\hat{K}_{\mathcal{A}} = (\kappa_{\mathcal{A}}, \xi_B): \widehat{\mathcal{BA}}(\mathcal{U}(\text{Lie}(\mathcal{A}))) \rightarrow \mathcal{A}$, where $\xi_B: \mathcal{U}(\text{Lie}(B)) \rightarrow B$ is the usual counit, and $\kappa_{\mathcal{A}}(a \otimes x) = \xi_A^b(a) \triangleleft \xi_B(x)$, where $a \in \mathcal{U}^0(\text{Lie}(A))$ and $x \in \mathcal{U}(\text{Lie}(B))$. (Note that A need not have a unit). Given that the second Peiffer relations are satisfied by \mathcal{A} , $\hat{K}_{\mathcal{A}}$ descends to a map $K_{\mathcal{A}}: \mathcal{BA}(\text{Lie}(\mathcal{A})) \rightarrow \mathcal{A}$, of crossed modules of bare algebras, called the bare counit. It defines (see below) a natural transformation of functors $K: \mathcal{BA} \circ \mathcal{U} \circ \text{Lie} \Rightarrow \text{id}$, the latter being the identity functor of the category of unital crossed modules of bare algebras. The natural transformation K is the counit of the adjunction of diagram (11):

$$\begin{array}{ccc}
 & \text{Lie} & \\
 & \top & \\
 \{\text{Lie algebras C.M.}\} & \xrightarrow{\quad K \quad} & \{\text{unital bare algebras C.M.}\} \\
 \mathcal{U} \left(\begin{array}{c} \uparrow \\ \{\text{Hopf Algebras C.M.}\} \end{array} \right. & \xrightarrow{\quad \mathcal{BA} \quad} &
 \end{array}
 \tag{12}$$

Given a unital crossed module of bare algebras $\mathcal{A} = (\partial: A \rightarrow B, \triangleright, \triangleleft)$, the pair of algebra maps $(\xi_A^{\#}: \mathcal{U}(\text{Lie}(A)) \rightarrow A_{\bullet}, \xi_B: \mathcal{U}(\text{Lie}(B)) \rightarrow B)$ yields a group crossed module map $\text{Proj}_{\mathcal{A}}: (\mathcal{U}(\text{Lie}(\mathcal{A}))_{\text{gl}}^*) \rightarrow (\mathcal{A})_{\bullet}^*$, called the crossed module counit. This yields a natural transformation of functors, as below:

$$\begin{array}{ccc}
 & \text{Lie} & \\
 & \top & \\
 \{\text{Lie algebras C.M.}\} & \xrightarrow{\quad \text{Proj} \quad} & \{\text{unital bare algebras C.M.}\} \\
 \mathcal{U} \left(\begin{array}{c} \uparrow \\ \{\text{Hopf Algebras C.M.}\} \end{array} \right. & \xrightarrow{\quad \text{Proj} \quad} & \\
 & \xrightarrow{\quad (\cdot)_{\text{gl}}^* \quad} & \xrightarrow{\quad (\cdot)_{\bullet}^* \quad} \\
 & \{\text{Groups C.M.}\} &
 \end{array}
 \tag{13}$$

The crossed module counit $\text{Proj}_{\mathcal{A}}: (\mathcal{U}(\text{Lie}(\mathcal{A}))_{\text{gl}}^* \rightarrow (\mathcal{A})_{\bullet}^*$, is the composition of the natural transformation in (12) with the natural transformation $\text{Inc} = (J, \iota): (\cdot)_{\bullet}^* \circ \mathcal{BA} \Rightarrow (\cdot)_{\text{gl}}^*$, the crossed module inclusion; (14).

$$\begin{array}{ccc}
 & & \{\text{unital bare algebras C.M.}\} \\
 & \xrightarrow{\quad \mathcal{BA} \quad} & \\
 \{\text{Hopf Algebras C.M.}\} & \xrightarrow{\quad \text{Inc} \quad} & \\
 & \xrightarrow{\quad (\cdot)_{\text{gl}}^* \quad} & \xrightarrow{\quad (\cdot)_{\bullet}^* \quad} \\
 & \{\text{Groups C.M.}\} &
 \end{array}
 \tag{14}$$

Given a crossed module $\mathcal{H} = (\partial: I \rightarrow H, \rho)$, then $\iota_{\mathcal{H}}: H_{\text{gl}}^* \rightarrow H^*$ is simply group inclusion. But not for $J_{\mathcal{H}}$.

In the bottom right corner of the diagram (10), we have the usual functor \mathcal{C}^{\times} , mentioned in the beginning of the Introduction, from the category of group crossed modules to the category of 2-groupoids; [BHS11]. If

$\mathcal{X} = (\partial: E \rightarrow G, \triangleright)$ is a crossed module of groups, all compositions in the 2-groupoid $\mathcal{C}^\times(\mathcal{A})$ are induced by the group multiplications of E and G . We can also go directly from the category of unital crossed modules of bare algebras to the category of 2-groupoids, through a functor $\mathcal{C}^+(\cdot)$, which, given a unital crossed module of bare algebras $\mathcal{A} = (\partial: A \rightarrow B, \triangleright, \triangleleft)$, uses the sum in A to define the vertical composition of 2-morphisms. What is far from obvious is that there exists also a natural transformation \mathcal{T} connecting the functor $\mathcal{C}^\times(\cdot) \circ (\cdot)^*$ to the functor $\mathcal{C}^+(\cdot)$; see the diagram below.

$$\begin{array}{ccc}
 & \{\text{unital bare algebras C.M.}\} & (15) \\
 & \swarrow (\cdot)^* & \downarrow \mathcal{C}^+(\cdot) \\
 \{\text{Groups C.M.}\} & \xrightarrow{\mathcal{C}^\times(\cdot)} & \{\text{2-groupoids}\} \\
 & \searrow \mathcal{T} &
 \end{array}$$

We note that a pre-crossed module $\hat{\mathcal{A}}$ of bare algebras also defines a sesquigroupoid $\hat{\mathcal{C}}^+(\hat{\mathcal{A}})$, the latter being a 2-groupoid if, and only if, $\hat{\mathcal{A}}$ is a crossed module. A sesquigroupoid [Str96] is defined exactly like a 2-groupoid, except that we do not impose the interchange law to hold, and therefore the horizontal composition of 2-morphisms in $\hat{\mathcal{C}}^+(\hat{\mathcal{A}})$ is not necessarily well defined.

1.1.3 Main differential geometric results

Let M be a differential manifold. Let $\mathcal{P}(M) = (\mathcal{P}^2(M), \mathcal{P}^1(M), M)$ be disjoint union of the sets of (piecewise smooth) 2-paths $\Gamma: [0, 1]^2 \rightarrow M$, 1-paths $\gamma: [0, 1] \rightarrow M$, as well as the underlying set of M . There are obvious vertical and horizontal composition (partial) operation in $\mathcal{P}^2(M)$, as well as vertical and horizontal reverses, composition of paths in $\mathcal{P}^1(M)$, and reverses, and also two whiskering (left and right) of 2-paths by 1-paths. We additionally consider several, however obvious, boundary and identity maps. The main results of this paper as far as two-dimensional holonomy is concerned are:

The Exact Holonomy $\mathcal{F}_{(\bar{\omega}, \bar{m})}^\times$: Let $\mathcal{H} = (\partial: I \rightarrow H, \rho)$ be a crossed module of Hopf algebras. Let (ω, m_1, m_2) be a fully primitive Hopf 2-connection over \mathcal{H} . This means that $\omega \in \Omega^1(M, H)$ is a 1-form in M , with values in H , and $m_1, m_2 \in \Omega^2(M, I)$ are 2-forms in M , with values in I , compatible in the sense that $\partial(m_1) = d\omega$ and $\partial(m_2) = -\frac{1}{2}\omega \wedge \omega$. Considering the graded forms $\bar{\omega} = h\omega$ and $\bar{m} = hm_1 + h^2m_2$, the compatibility condition is equivalent to imposing that $\partial(\bar{m}) = \mathcal{F}_{\bar{\omega}} = hd\omega - h^2\frac{1}{2}\omega \wedge \omega$. The prefix “fully primitive” means that both ω, m_1 and m_2 take values in the primitive spaces of H and I . We have a map $\mathcal{F}_{(\bar{\omega}, \bar{m})}^\times: \mathcal{P}(M) \rightarrow \mathcal{C}^\times(\mathcal{H}_{\text{gl}}^*)$, the latter being a 2-groupoid, preserving all compositions, boundaries and reverses, in $\mathcal{P}(M)$ and in $\mathcal{C}^\times(\mathcal{H}_{\text{gl}}^*)$. We call $\mathcal{F}_{(\bar{\omega}, \bar{m})}^\times$ the “Exact Holonomy”.

The Bare Holonomy $\mathcal{F}_{(\bar{\omega}, \bar{m})}^+$: Let $\hat{\mathcal{A}} = (\partial: A \rightarrow B, \triangleright, \triangleleft)$, be a pre-crossed module of bare algebras, where $\triangleright, \triangleleft$ are left and right actions of B on A , by multipliers. Let (ω, m_1, m_2) be a bare 2-connection over $\hat{\mathcal{A}}$. This means that $\omega \in \Omega^1(M, B)$ and $m_1, m_2 \in \Omega^2(M, A)$ are such that, if we put $\bar{\omega} = h\omega$ and $\bar{m} = hm_1 + h^2m_2$, it holds that $\partial(\bar{m}) = \mathcal{F}_{\bar{\omega}} = hd\omega - h^2\frac{1}{2}\omega \wedge \omega$. We have a map $\mathcal{F}_{(\bar{\omega}, \bar{m})}^+: \mathcal{P}(M) \rightarrow \hat{\mathcal{C}}^+(\hat{\mathcal{A}})$, preserving all boundaries, composition of 1-morphisms, vertical reverses and composition of 2-morphisms, but not necessarily the horizontal composition of 2-morphisms (which is not even well defined in $\hat{\mathcal{C}}^+(\hat{\mathcal{A}})$, unless $\hat{\mathcal{A}}$ is a crossed module, of bare algebras.) If $\hat{\mathcal{A}}$ is a crossed module, thus $\hat{\mathcal{C}}^+(\hat{\mathcal{A}}) = \mathcal{C}^+(\hat{\mathcal{A}})$ is a 2-groupoid, then the map $\mathcal{F}_{(\bar{\omega}, \bar{m})}^+: \mathcal{P}(M) \rightarrow \mathcal{C}^+(\hat{\mathcal{A}})$ also preserves horizontal composition (and reverses) of 2-morphisms, in $\mathcal{P}(M)$ and in $\mathcal{C}^+(\hat{\mathcal{A}})$.

The Fuzzy Holonomy $\mathcal{Z}_{(\bar{\omega}, \bar{m})}^+$ and the Blur Holonomy $\hat{\mathcal{Z}}_{(\bar{\omega}, \bar{m})}^+$ Let $\mathcal{H} = (\partial: I \rightarrow H, \rho)$ be a crossed module of Hopf algebras. Let (ω, m_1, m_2) be a fully primitive Hopf 2-connection over \mathcal{H} . We can see (ω, m_1, m_2) as being a bare 2-connection with values in $\widehat{\mathcal{BA}}(\mathcal{H}) = (t: I^0 \otimes_\rho H \rightarrow H, \triangleright, \triangleleft)$, a pre-crossed module of bare algebras, or, projecting into $\mathcal{BA}(\mathcal{H}) = (t: \underline{I^0 \otimes_\rho H} \rightarrow H, \triangleright, \triangleleft)$, as a bare 2-connection with values in the crossed module $\mathcal{BA}(\mathcal{H})$. The two dimensional holonomy of the former is called the blur holonomy, and is denoted by $\hat{\mathcal{Z}}_{(\bar{\omega}, \bar{m})}^+: \mathcal{P}(M) \rightarrow \hat{\mathcal{C}}^+(\widehat{\mathcal{BA}}(\mathcal{H}))$. The two dimensional holonomy of the latter is called the fuzzy holonomy and is denoted by: $\mathcal{Z}_{(\bar{\omega}, \bar{m})}^+: \mathcal{P}(M) \rightarrow \mathcal{C}^+(\mathcal{BA}(\mathcal{H}))$. By construction, and the item above, the fuzzy holonomy preserves all compositions and reverses. The blur holonomy does not, necessarily, preserve horizontal compositions (and reverses) of 2-morphisms.

Compatibility between formulations of two dimensional holonomy-I This is a non-trivial relation between the bare and exact formulations of two dimensional holonomy (a relation between the exact and the seemingly much finer blur holonomy remains elusive). Let $\mathcal{H} = (\partial: I \rightarrow H, \rho)$ be a crossed module of Hopf algebras. Let (ω, m_1, m_2) be a fully primitive Hopf 2-connection over \mathcal{H} . The fuzzy and exact holonomies of (ω, m_1, m_2) are compatible, in the sense that the following diagram commutes (recall the natural transformations Inc and \mathcal{T} of diagrams (14) and (15)):

$$\begin{array}{ccc}
\mathcal{C}^\times \left((\mathcal{H})_{\text{gl}}^* \right) & \xleftarrow{\mathcal{F}_{(\omega, \overline{m})}^\times} & \mathcal{P}(M) \\
\downarrow \mathcal{C}^\times(\text{Inc}_{\mathcal{H}}) & & \downarrow \mathcal{Z}_{(\omega, \overline{m})}^+ \\
\mathcal{C}^\times \left(\mathcal{BA}(\mathcal{H})_{\bullet}^* \right) & \xrightarrow{\mathcal{T}_{\mathcal{BA}(\mathcal{H})}} & \mathcal{C}^+ \left(\mathcal{BA}(\mathcal{H}) \right)
\end{array} \tag{16}$$

Compatibility between formulations of two dimensional holonomy-II Let $\mathcal{A} = (\partial: A \rightarrow B, \triangleright, \triangleleft)$, be a crossed module of bare algebras. Let (ω, m_1, m_2) be a bare 2-connection over \mathcal{A} . We can also see (ω, m_1, m_2) as being a fully primitive Hopf 2-connection over $\mathcal{U}(\text{Lie}(\mathcal{A}))$, and consequently also as a bare 2-connection over $\mathcal{BA}(\mathcal{U}(\text{Lie}(\mathcal{A})))$. Recall (10),(12), (13) and (14). The diagram commutes:

$$\begin{array}{ccccc}
& \mathcal{C}^\times \left(\mathcal{U}(\text{Lie}(\mathcal{A}))_{\text{gl}}^* \right) & \xleftarrow{\mathcal{F}_{(\omega, \overline{m})}^\times} & \mathcal{P}(M) & \\
& \downarrow \mathcal{C}^\times(\text{Inc}_{\mathcal{U}(\text{Lie}(\mathcal{A}))}) & & \downarrow \mathcal{Z}_{(\omega, \overline{m})}^+ & \\
\mathcal{C}^\times(\text{Proj}_{\mathcal{A}}) \nearrow & \mathcal{C}^\times \left(\mathcal{BA}(\mathcal{U}(\text{Lie}(\mathcal{A})))_{\bullet}^* \right) & \xrightarrow{\mathcal{T}_{\mathcal{BA}(\mathcal{U}(\text{Lie}(\mathcal{A})))}} & \mathcal{C}^+ \left(\mathcal{BA}(\mathcal{U}(\text{Lie}(\mathcal{A}))) \right) & \xrightarrow{\mathcal{C}^+(K_{\mathcal{A}})} & \mathcal{C}^+(\mathcal{A}) \\
& \swarrow \mathcal{C}^\times((K_{\mathcal{A}})_{\bullet}^*) & & & & \\
\mathcal{C}^\times(\mathcal{A}_{\bullet}^*) & & \xrightarrow{\mathcal{T}_{\mathcal{A}}} & & &
\end{array} \tag{17}$$

(This essentially follows from the previous diagram (16), either by definition, functorially or naturally.)

If we unpack what the commutativity of the outer polygon of (17) means, in the particular case of crossed modules derived from chain-complexes of vector spaces, we obtain an equality between two versions of two dimensional holonomy (of 2-connections and of representations up to homotopy), previously independently noticed in [CFM12a, AS14a], being fully proved in the latter reference.

2 Algebraic preliminaries

All algebras will by default be over \mathbb{C} , sometimes over $\mathbb{C}[[h]]$, the latter always being explicitly mentioned.

2.1 Crossed modules of groups and of Lie algebras

Definition 1 ((Pre-) Crossed module of (Lie) groups). *A modern treatment of group crossed modules appears in [Bau91, BL04, BHS11, Bro99]. A crossed module (of groups) $\mathcal{X} = (\partial: E \rightarrow G, \triangleright)$ is given by a group morphism $\partial: E \rightarrow G$, together with a left action \triangleright of G on E by automorphisms, such that:*

1. $\partial(g \triangleright e) = g \partial(e) g^{-1}$; for each $g \in G$ and $e \in E$, (First Peiffer relation).
2. $\partial(e) \triangleright e' = e e' e^{-1}$; for each $e, e' \in E$. (Second Peiffer relation).

If both G and E are Lie groups, $\partial: E \rightarrow G$ is a smooth morphism, and the left action of G on E is smooth, then \mathcal{X} will be called a Lie crossed module. If $\hat{\mathcal{X}}$ satisfies all relations for it to be a group crossed module, except possibly for the second Peiffer relation, then $\hat{\mathcal{X}} = (\partial: E \rightarrow G, \triangleright)$ will be called a pre-crossed module.

A morphism $(\psi, \phi): \mathcal{X} \rightarrow \mathcal{X}'$ between the crossed modules $\mathcal{X} = (\partial: H \rightarrow G, \triangleright)$ and $\mathcal{X}' = (\partial': H' \rightarrow G', \triangleright')$ is given by a pair of group morphisms $(\psi: H \rightarrow H', \phi: G \rightarrow G')$, such that the diagram:

$$\begin{array}{ccc} H & \xrightarrow{\partial} & G \\ \psi \downarrow & & \downarrow \phi \\ H' & \xrightarrow{\partial'} & G' \end{array}$$

is commutative. In addition, we must have $\psi(g \triangleright h) = \phi(g) \triangleright' \psi(h)$, for each $h \in H$ and each $g \in G$.

Given a Lie crossed module $\mathcal{X} = (\partial: H \rightarrow G, \triangleright)$, then the induced Lie algebra map $\partial: \mathfrak{e} \rightarrow \mathfrak{g}$, together with the derived action of \mathfrak{g} on \mathfrak{e} (also denoted by \triangleright) is a differential crossed module, in the sense below:

Definition 2 (Differential (pre-) crossed module). *A differential crossed module $\mathfrak{X} = (\partial: \mathfrak{e} \rightarrow \mathfrak{g}, \triangleright)$, see [BL04, BC04, Wag06], also called crossed module of Lie algebras, is given by a Lie algebra map $\partial: \mathfrak{e} \rightarrow \mathfrak{g}$, together with a left action (by derivations, see below) of \mathfrak{g} on the underlying vector space of \mathfrak{e} , such that:*

1. For any $X \in \mathfrak{g}$ the map $\zeta \in \mathfrak{e} \mapsto X \triangleright \zeta \in \mathfrak{e}$ is a derivation of \mathfrak{e} , in other words:

$$X \triangleright [\zeta, \nu] = [X \triangleright \zeta, \nu] + [\zeta, X \triangleright \nu], \quad \forall X \in \mathfrak{g}, \forall \zeta, \nu \in \mathfrak{e}.$$

2. The map $\mathfrak{g} \rightarrow \text{Der}(\mathfrak{e})$, from \mathfrak{g} into the derivation algebra of \mathfrak{e} , induced by the action of \mathfrak{g} on \mathfrak{e} , is a Lie algebra morphism, in other words:

$$[X, Y] \triangleright \zeta = X \triangleright (Y \triangleright \zeta) - Y \triangleright (X \triangleright \zeta), \quad \forall X, Y \in \mathfrak{g}, \forall \zeta \in \mathfrak{e},$$

3. $\partial(X \triangleright \zeta) = [X, \partial(\zeta)]$, $\forall X \in \mathfrak{g}, \forall \zeta \in \mathfrak{e}$, (First Peiffer relation).

4. $\partial(\zeta) \triangleright \nu = [\zeta, \nu]$, $\forall \zeta, \nu \in \mathfrak{e}$, (Second Peiffer relation).

If $\hat{\mathfrak{X}} = (\partial: \mathfrak{e} \rightarrow \mathfrak{g}, \triangleright)$ satisfies all conditions for it to be a differential crossed module, except possibly for the second Peiffer relation, then $\hat{\mathfrak{X}} = (\partial: \mathfrak{e} \rightarrow \mathfrak{g}, \triangleright)$ will be called a differential pre-crossed module.

Given a differential crossed module $\mathfrak{X} = (\partial: \mathfrak{e} \rightarrow \mathfrak{g}, \triangleright)$, where \mathfrak{e} and \mathfrak{g} are finite dimensional, there exists a unique, up to isomorphism, crossed module of simply connected Lie groups $\mathcal{X} = (\partial: H \rightarrow G, \triangleright)$ whose differential form is \mathfrak{X} .

2.2 Crossed modules of bare algebras

2.2.1 Definition of crossed modules and pre-crossed modules of bare algebras

Definition 3 (Bare algebra). *A bare algebra A (as opposed to a bi-algebra, a unital algebra, or a Hopf algebra) is an associative algebra (not necessarily abelian) over a field (in this paper always \mathbb{C}) with or without identity, and without taking into account any other structure which may possibly exist in A .*

We propose the following definition of a crossed module of bare algebras. This is a working definition, completely bypassing its (unknown, and possibly not-existing) relation with simplicial algebras with Moore complex of length one. We note that crossed modules of associative algebras are defined in [Arv04, CIKL14, DIKL12], and our definition is more restrictive. Notice that, in the case of commutative algebras (where left and right actions are the same thing) our definition of crossed modules coincides with the one in [AP96].

We recall that if A is a bare algebra, then a left multiplier of A is a linear map $f: A \rightarrow A$ such that $f(ab) = f(a)b$, for each $a, b \in A$. A right multiplier is a linear map $g: A \rightarrow A$ such that $g(ab) = ag(b)$, for each $a, b \in A$. The composition of linear maps $A \rightarrow A$ gives the algebras of left and right multipliers of A .

Definition 4 (Crossed modules of bare algebras). *A crossed module $\mathcal{A} = (\partial: A \rightarrow B, \triangleright, \triangleleft)$ of bare algebras, is given by an algebra map $\partial: A \rightarrow B$, together with two bilinear maps:*

$$(a, b) \in A \times B \mapsto a \triangleleft b \in A, \quad (b, a) \in B \times A \mapsto b \triangleright a \in A.$$

The remaining conditions are:

- \triangleleft and \triangleright are right and left actions of algebras on vector spaces, namely we have:

$$(bb') \triangleright a = b \triangleright (b' \triangleright a) \text{ and } a \triangleleft (bb') = (a \triangleleft b) \triangleleft b', \text{ for each } b, b' \in B \text{ and } a \in A.$$

- \triangleleft and \triangleright are (right and left) actions by (left and right multipliers) and therefore are compatible with the multiplication in A :

$$b \triangleright (aa') = (b \triangleright a)a' \text{ and } (aa') \triangleleft b = a(a' \triangleleft b), \text{ for each } a, a' \in A \text{ and } b \in B.$$

- The left and right actions commute:

$$b \triangleright (a \triangleleft b') = (b \triangleright a) \triangleleft b', \text{ for each } b, b' \in B \text{ and } a \in A.$$

- The left and right actions are fully interchangeable, in the sense that:

$$(a \triangleleft b) (b' \triangleright a') = a ((bb') \triangleright a') = (a \triangleleft (bb')) a', \text{ for each } b, b' \in B \text{ and } a, a' \in A.$$

- The first Peiffer relations hold:

$$\partial(b \triangleright a) = b\partial(a) \text{ and } \partial(a \triangleleft b) = \partial(a)b, \text{ for each } a \in A \text{ and } b \in B.$$

- The second Peiffer relations hold:

$$\partial(a') \triangleright a = a'a \text{ and } a \triangleleft \partial(a') = aa', \text{ for each } a, a' \in A.$$

As it has been customary, if $\hat{A} = (\partial: A \rightarrow B, \triangleright, \triangleleft)$ is such that all conditions for it to be a crossed module of bare algebras hold, except possibly for the second Peiffer relation, then $\hat{A} = (\partial: A \rightarrow B, \triangleright, \triangleleft)$ will be called a pre-crossed module, of bare algebras.

Definition 5 (Unital crossed module of bare algebras). A crossed (or pre-crossed) module $\mathcal{A} = (\partial: A \rightarrow B, \triangleright, \triangleleft)$ of bare algebras is said to be unital, if B has an identity 1_B , satisfying $1_B \triangleright a = a = a \triangleleft 1_B, \forall a \in A$.

2.2.2 From pre-crossed modules of bare algebras to crossed modules of bare algebras

A useful identity holding in any crossed module of bare algebras $\mathcal{A} = (\partial: A \rightarrow B, \triangleright, \triangleleft)$ is the following:

$$\partial(a) \triangleright a' = aa' = a \triangleleft \partial(a'), \text{ for each } a, a' \in A. \quad (18)$$

This fails to be true, in general, in a pre-crossed module. Given a pre-crossed module $\hat{A} = (\partial: \hat{A} \rightarrow B, \triangleright, \triangleleft)$, of bare algebras, consider the following bilinear map in A , vanishing in the crossed module case:

$$(a, a') \in A \times A \mapsto \{a, a'\} \doteq \partial(a) \triangleright a' - a \triangleleft \partial(a') \in A. \quad (19)$$

Let L be the biideal of A generated by the image of the linear map $a \otimes a' \in A \otimes A \mapsto \{a, a'\}$. The pre-crossed module axioms imply that L stays invariant under the actions \triangleright and \triangleleft of B on A . Put $\underline{A} = A/L$. Then the actions \triangleright and \triangleleft descend to actions \triangleright and \triangleleft of B on \underline{A} , by multipliers. By using the first Peiffer relation $\partial: A \rightarrow B$ descends to a map $\partial: \underline{A} \rightarrow B$. Clearly $(\partial: \underline{A} \rightarrow B, \triangleright, \triangleleft) \doteq \mathcal{R}((\partial: A \rightarrow B, \triangleright, \triangleleft))$ is a crossed module of bare algebras. Moreover we have a functor \mathcal{R} , from the category of pre-crossed modules of bare algebras, to the category of crossed modules of bare algebras, making the latter a reflexive subcategory.

2.2.3 From crossed modules of bare algebras to differential crossed modules

The following generalizes the construction of a Lie algebra $\text{Lie}(A)$ as being the commutator algebra of a bare algebra A , namely $[a, b] = ab - ba, \forall a, b \in A$. It follows from simple calculations. The fact that the left and right actions are fully interchangeable has a primary role in the proof that the action \triangleright , below, is by derivations. That \triangleright is itself an action uses solely the fact that \triangleright and \triangleleft commute.

Lemma 6. Let $\mathcal{A} = (\partial: A \rightarrow B, \triangleright, \triangleleft)$ be a crossed module of bare algebras. Then, looking at the associated Lie algebras $\text{Lie}(A)$ and $\text{Lie}(B)$, and the induced Lie algebra map $\partial: \text{Lie}(A) \rightarrow \text{Lie}(B)$, the bilinear map:

$$(b, a) \in B \otimes A \mapsto b \triangleright a \doteq b \triangleright a - a \triangleleft b \in A,$$

is a Lie algebra action of B on A by derivations, and, moreover, $\text{Lie}(\mathcal{A}) = (\partial: \text{Lie}(A) \rightarrow \text{Lie}(B), \triangleright)$ is a crossed module of Lie algebras.

2.2.4 From unital crossed modules of bare algebras to group crossed modules

There exists a functor from the category of unital crossed modules of bare algebras $\mathcal{A} = (\partial: A \rightarrow B, \triangleright, \triangleleft)$, to the category of crossed modules of groups. In the finite dimensional case, this is related to the composition of the functors, sending a crossed module of bare algebras to a differential crossed module, and sending a differential crossed module to a crossed module of simply connected Lie groups. In order to work with explicit models, and in the infinite dimensional case, we will do all constructions mostly over $\mathbb{C}[[h]]$ (except for the initial construction, only done for motivating purposes), making this relation appear less clear

Definition 7 (From A to A_\bullet : adjoining an identity). *Let A be bare algebra. We denote by A_\bullet the algebra derived from A by adjoining an identity, which is different from an eventually existing identity of A . Explicitly $A_\bullet = \mathbb{C} \oplus A$, with product $(\lambda, a)(\lambda', a') = (\lambda\lambda', \lambda a' + \lambda' a + aa')$, where $\lambda, \lambda' \in \mathbb{C}$ and $a, a' \in A$. We note that adjoining an identity is functorial, being left adjoint to the forgetful functor that assigns a bare algebra to an algebra with identity. (In the category of unital algebras morphisms are assumed to preserve identities.)*

Let $\mathcal{A} = (\partial: A \rightarrow B, \triangleright, \triangleleft)$ be a unital crossed module of bare algebras. There is a morphism of unital algebras $\partial: A_\bullet \rightarrow B$, where $\partial(\lambda, a) = \lambda 1_B + \partial(a)$, for each $\lambda \in \mathbb{C}$ and $a \in A$. Let $\text{Units}(B)$ denote the group of invertible elements (units) of B . The following follows easily from the definition of a crossed module of bare algebras. The fact that \triangleleft and \triangleright are fully interchangeable plays a fundamental role in proving that the action of $\text{Units}(B)$ on A_\bullet is by algebra isomorphisms.

Proposition 8. *The map $(b, (\lambda, a)) \in \text{Units}(B) \times A_\bullet \mapsto b \triangleright (\lambda, a) \doteq (\lambda, b \triangleright a \triangleleft b^{-1})$ is an action of the group $\text{Units}(B)$ on A_\bullet by unital algebra isomorphisms. Moreover, if we pass to the group $\text{Units}(A_\bullet)$ of units of A_\bullet , then $\text{Units}(A) \doteq (\partial: \text{Units}(A_\bullet) \rightarrow \text{Units}(B), \triangleright)$ is a crossed module of groups.*

Proof: The only unclear bit is the second Peiffer relation. Let (λ, a) and (λ', a') be invertible elements of A_\bullet . We want to prove that $\partial(\lambda, a) \triangleright (\lambda', a') = (\lambda, a) (\lambda', a') (\lambda, a)^{-1}$. We have, putting $(\lambda, a)^{-1} = (\lambda^{-1}, \bar{a})$:

$$\begin{aligned} \partial(\lambda, a) \triangleright (\lambda', a') &= (\lambda', \partial(\lambda, a) \triangleright a' \triangleleft \partial(\lambda, a)^{-1}) = (\lambda', (\lambda 1_B + \partial(a)) \triangleright a' \triangleleft (\lambda^{-1} 1_B + \partial(\bar{a}))) \\ &= (\lambda', a' + \lambda^{-1} \partial(a) \triangleright a' + \lambda a' \triangleleft \partial(\bar{a}) + \partial(a) \triangleright a' \triangleleft \partial(\bar{a})). \end{aligned}$$

Whereas:

$$\begin{aligned} (\lambda, a) (\lambda', a') (\lambda, a)^{-1} &= (\lambda\lambda', \lambda' a + \lambda a' + aa') (\lambda^{-1}, \bar{a}) \\ &= (\lambda', \lambda^{-1} \lambda' a + \lambda^{-1} \lambda a' + \lambda^{-1} aa' + \lambda' a \bar{a} + \lambda a' \bar{a} + aa' \bar{a} + \lambda \lambda' \bar{a}). \end{aligned}$$

By using the second Peiffer relation for crossed module of bare algebras, we have:

$$\partial(\lambda, a) \triangleright (\lambda', a') - (\lambda, a) (\lambda', a') (\lambda, a)^{-1} = -(0, \lambda^{-1} \lambda' a + \lambda' a \bar{a} + \lambda \lambda' \bar{a}) = (0, 0),$$

since we have $(1, 0) = (\lambda, a) (\lambda^{-1}, \bar{a}) = (1, a \bar{a} + \lambda \bar{a} + \lambda^{-1} a)$. \square

For this paper we will only need a (slightly restricted) version of the previous construction, however over $\mathbb{C}[[h]]$, by adjoining a formal parameter h to our construction. Recalling that A is a bare algebra, we let $A[[h]]$ be the $\mathbb{C}[[h]]$ algebra of formal power series over A , with the obvious product, the $\mathbb{C}[[h]]$ -linear extension of the product over A . We let $A_\bullet[[h]]$ be the algebra obtained from $A[[h]]$ by (functorially) adjoining an identity element. Therefore $A_\bullet[[h]] = \mathbb{C}[[h]] \oplus A[[h]]$, as a $\mathbb{C}[[h]]$ -module, with product: $(p, a)(p', a') = (pp', pa' + p'a + aa')$, where $p, p' \in \mathbb{C}[[h]]$ and $a, a' \in A[[h]]$. The identity is: $1 = (1 + 0h + 0h^2 + 0h^3 + \dots, 0)$.

Recall that any power series, over an algebra, whose first element is the identity is invertible. Let $A_\bullet^* \subsetneq \text{Units}(A_\bullet[[h]])$ be the group of formal power series in A_\bullet , whose first term is $1_{\mathbb{C}} = (1_{\mathbb{C}}, 0)$, and such that all other terms are h^n multiples of elements of $A = \{0\} \times A$. Write each element in A_\bullet^* as $a = 1 + \sum_{n=1}^{+\infty} a_n h^n$, where $a_n \in A, \forall n$. Let $\mathcal{A} = (\partial: A \rightarrow B, \triangleleft, \triangleright)$ be a unital crossed module of bare algebras. Let $B^* \subsetneq \text{Units}(B[[h]])$ be the group of formal power series over B of the form $b = 1_B + \sum_{n=1}^{+\infty} h^n b_n$, where $b_n \in B, \forall n$. Consider the actions \triangleleft and \triangleright of $B[[h]]$ on $A[[h]]$ obtained by extending \triangleleft and \triangleright $\mathbb{C}[[h]]$ -linearly.

Proposition 9. *Let $\mathcal{A} = (\partial: A \rightarrow B, \triangleright, \triangleleft)$ be a crossed module of bare algebras, with B unital. Consider the induced $\mathbb{C}[[h]]$ -algebra map $\partial: A_\bullet[[h]] \rightarrow B[[h]]$, restricting to a group map $\partial: A_\bullet^* \rightarrow B^*$. With the left action $b \triangleright (1, a) \doteq (1, b \triangleright a \triangleleft b^{-1}), \forall b \in B^*, \forall a \in A[[h]]$ we have a crossed module of groups $\mathcal{A}_\bullet^* = (\partial: A_\bullet^* \rightarrow B^*, \triangleright)$.*

2.2.5 Crossed modules of bare algebras from chain complexes of vector spaces

The most suggestive non-trivial class of examples of unital crossed modules of bare algebras, comes from considering chain complexes $\mathcal{V} = (\dots \xrightarrow{\beta} V_n \xrightarrow{\beta} V_{n-1} \xrightarrow{\beta} \dots)$, of vector spaces. We let $\mathfrak{gl}_0(\mathcal{V})$ be the unital algebra of chain maps $f: \mathcal{V} \rightarrow \mathcal{V}$, with the usual composition of chain maps, and commutator $\{f, g\} = f \circ g - g \circ f$, where $f, g \in \mathfrak{gl}_0(\mathcal{V})$. We let $\text{hom}^i(\mathcal{V})$ denote the vector space of degree i maps $\mathcal{V} \rightarrow \mathcal{V}$. Composition yields a bilinear map $\text{hom}^i(\mathcal{V}) \times \text{hom}^j(\mathcal{V}) \rightarrow \text{hom}^{i+j}(\mathcal{V})$. A degree-1 map $s: \mathcal{V} \rightarrow \mathcal{V}$, will be called a homotopy, and is given by a sequence of vector space maps $\{s_i: V_i \rightarrow V_{i+1}\}_{i \in \mathbb{Z}}$, with no compatibility relation with respect to the boundary maps β . There is a map $\partial': \text{hom}^1(\mathcal{V}) \rightarrow \mathfrak{gl}_0(\mathcal{V}) \subset \text{hom}^0(\mathcal{V})$ with $\partial'(s) = \beta s + s\beta, \forall s \in \text{hom}^1(\mathcal{V})$. If $f \in \mathfrak{gl}_0(\mathcal{V})$ and $s \in \text{hom}^1(\mathcal{V})$:

$$\partial'(f \circ s) = f \circ \partial'(s), \quad \partial'(s \circ f) = \partial'(s) \circ f. \quad (20)$$

A degree-2 map $h: \mathcal{V} \rightarrow \mathcal{V}$ is a sequence of vector space maps $\{h_i: V_i \rightarrow V_{i+2}\}_{i \in \mathbb{Z}}$. There is a map $\partial': \text{hom}^2(\mathcal{V}) \rightarrow \text{hom}^1(\mathcal{V})$, with $\partial'(h) = \beta h - h\beta, \forall h \in \text{hom}^2(\mathcal{V})$. If $f \in \mathfrak{gl}_0(\mathcal{V})$ and $h \in \text{hom}^2(\mathcal{V})$, then:

$$\partial'(f \circ h) = f \circ \partial'(h), \quad \partial'(h \circ f) = \partial'(h) \circ f. \quad (21)$$

Put $\mathfrak{gl}_1(\mathcal{V}) = \text{hom}^1(\mathcal{V})/\partial'(\text{hom}^2(\mathcal{V}))$. Then clearly $\partial': \text{hom}^1(\mathcal{V}) \rightarrow \mathfrak{gl}_0(\mathcal{V})$ descends to a linear map $\partial: \mathfrak{gl}_1(\mathcal{V}) \rightarrow \mathfrak{gl}_0(\mathcal{V})$. Moreover the composition maps:

$$(f, s) \in \text{hom}_0(\mathcal{V}) \times \text{hom}_1(\mathcal{V}) \mapsto f \circ s \in \text{hom}_1(\mathcal{V}), \quad (s, f) \in \text{hom}_1(\mathcal{V}) \times \text{hom}_0(\mathcal{V}) \mapsto s \circ f \in \text{hom}_1(\mathcal{V})$$

descend to maps:

$$(f, s) \in \mathfrak{gl}_0(\mathcal{V}) \times \mathfrak{gl}_1(\mathcal{V}) \mapsto s \circ f \in \mathfrak{gl}_1(\mathcal{V}), \quad (s, f) \in \mathfrak{gl}_1(\mathcal{V}) \times \mathfrak{gl}_0(\mathcal{V}) \mapsto s \circ f \in \mathfrak{gl}_1(\mathcal{V}).$$

Lemma 10. *For each $s, t \in \text{hom}^1(\mathcal{V})$, the following holds:*

$$\partial'(s \circ t) = \partial'(s) \circ t - s \circ \partial'(t). \quad (22)$$

In particular in $\mathfrak{gl}_1(\mathcal{V})$ we have:

$$\partial(s) \circ t = s \circ \partial(t).$$

We consider the algebra structure in $\mathfrak{gl}_0(\mathcal{V})$ given by the composition of chain-maps. In $\mathfrak{gl}_1(\mathcal{V})$, we consider the following associative algebra structure:

$$s * t \doteq s \circ \partial(t) = \partial(s) \circ t.$$

Theorem 11. *Let $\mathcal{V} = (\dots \xrightarrow{\beta} V_n \xrightarrow{\beta} V_{n-1} \xrightarrow{\beta} \dots)$ be a chain complex of vector spaces. Then $\partial: \mathfrak{gl}_1(\mathcal{V}) \rightarrow \mathfrak{gl}_0(\mathcal{V})$, with $\partial(s) = s\beta + \beta s, \forall s \in \mathfrak{gl}_1(\mathcal{V})$ is an algebra map. Moreover, with the left and right actions:*

$$\begin{aligned} (f, s) \in \mathfrak{gl}_0(\mathcal{V}) \times \mathfrak{gl}_1(\mathcal{V}) &\rightarrow \mathfrak{gl}_1(\mathcal{V}) \mapsto f \triangleright s \doteq f \circ s, \\ (s, f) \in \mathfrak{gl}_1(\mathcal{V}) \times \mathfrak{gl}_0(\mathcal{V}) &\rightarrow \mathfrak{gl}_1(\mathcal{V}) \mapsto s \triangleleft f \doteq s \circ f, \end{aligned}$$

this defines a unital crossed module of bare algebras, $\mathcal{HOM}(\mathcal{V}) = (\partial: \mathfrak{gl}_1(\mathcal{V}) \rightarrow \mathfrak{gl}_0(\mathcal{V}), \triangleright, \triangleleft)$.

Remark 12 (Crossed modules of groups and Lie algebras from chain-complexes). *Combining the previous theorem with 2.2.3 and 2.2.4, yields, given a chain complex \mathcal{V} of vector spaces, a differential crossed module:*

$$\mathfrak{GL}(\mathcal{V}) = (\partial: \mathfrak{gl}_1(\mathcal{V}) \rightarrow \mathfrak{gl}_0(\mathcal{V}), \triangleright) = \text{Lie}(\mathcal{HOM}(\mathcal{V}))$$

and group crossed modules, previously defined (essentially) in [BL04, KP02, FMM11, CFM12b, CFM12a]:

$$\begin{aligned} \text{GL}(\mathcal{V}) &= (\partial: \text{Units}(\mathfrak{gl}_1(\mathcal{V})) \rightarrow \text{Units}(\mathfrak{gl}_0(\mathcal{V})), \triangleright) = \text{Units}(\mathcal{HOM}(\mathcal{V})), \\ \text{GL}(\mathcal{V}, h) &= (\partial: \mathfrak{gl}_1(\mathcal{V})^* \rightarrow \mathfrak{gl}_0(\mathcal{V})^*, \triangleright) = \mathcal{HOM}(\mathcal{V})^*. \end{aligned}$$

2.3 Two-groupoids and crossed modules

2.3.1 The group crossed module case

For the definition of a 2-groupoid and of a sesquigroupoid see [HKK00, Mac98, Str96]. Let $\mathcal{X} = (\partial: E \rightarrow G, \triangleright)$ be a group crossed module. We can define a 2-groupoid $\mathcal{C}^\times(\mathcal{X})$ out of it, with a single object $*$, which is therefore a categorical group [BM06]. This is an old construction; see, for example, [BS76] and, more recently [BHS11, BL04]. We will take as being the “fundamental” compositions in $\mathcal{C}^\times(\mathcal{X})$ the vertical composition of 2-morphisms, as well as the whiskerings of 2-morphisms by 1-morphisms, as in [Str96, KP02], since the horizontal composition of 2-morphisms can be derived from these. The set of 1-morphisms of $\mathcal{C}^\times(\mathcal{X})$ is given by all arrows of the form $* \xrightarrow{g} *$, with $g \in G$, the composition being:

$$* \xrightarrow{g} * \xrightarrow{g'} * = * \xrightarrow{gg'} *$$

The set of 2-morphisms of $\mathcal{C}^\times(\mathcal{X})$ is given by all 2-arrows of the form:

$$* \begin{array}{c} \partial(e)^{-1}g \\ \curvearrowright \\ \boxed{\times} \uparrow e \\ \curvearrowleft \\ g \end{array} * , \text{ where } g \in G, \text{ and } e \in E,$$

each of these having vertical and horizontal inverses of the form (respectively):

$$* \begin{array}{c} g \\ \curvearrowright \\ \boxed{\times} \uparrow e^{-1} \\ \curvearrowleft \\ \partial(e)^{-1}g \end{array} * \quad \text{and} \quad * \begin{array}{c} g\partial(e) \\ \curvearrowright \\ \boxed{\times} \uparrow g^{-1} \triangleright e^{-1} \\ \curvearrowleft \\ g^{-1} \end{array} * .$$

The horizontal composition of 2-morphisms appears in (24). The vertical composition of 2-morphisms is:

$$\begin{array}{c} \partial(f)^{-1}\partial(e)^{-1}g \\ \curvearrowright \\ \boxed{\times} \uparrow f \\ \partial(e)^{-1}g \longrightarrow * \\ \boxed{\times} \uparrow e \\ \curvearrowleft \\ g \end{array} * = * \begin{array}{c} \partial(ef)^{-1}g \\ \curvearrowright \\ \boxed{\times} \uparrow ef \\ \curvearrowleft \\ g \end{array} * , \text{ where } g \in G, \text{ and } e, f \in E.$$

The left and right whiskerings are (for $g, h \in G$ and $e \in E$), respectively:

$$\begin{array}{c} \partial(e)^{-1}g \\ \curvearrowright \\ \boxed{\times} \uparrow e \\ \curvearrowleft \\ g \end{array} * \xrightarrow{h} * = * \begin{array}{c} \partial(e)^{-1}gh \\ \curvearrowright \\ \boxed{\times} \uparrow e \\ \curvearrowleft \\ gh \end{array} * ,$$

$$* \xrightarrow{h} * \begin{array}{c} \partial(e)^{-1}g \\ \curvearrowright \\ \boxed{\times} \uparrow e \\ \curvearrowleft \\ g \end{array} * = * \begin{array}{c} h\partial(e)^{-1}g \\ \curvearrowright \\ \boxed{\times} \uparrow h \triangleright e \\ \curvearrowleft \\ hg \end{array} * .$$

These whiskering maps are distributive with respect to the vertical composition of 2-morphisms, and are associative, whenever compositions are defined. In addition, the interchange law is satisfied, namely:

$$\begin{array}{c} \partial(e)^{-1}g \\ \curvearrowright \\ \boxed{\times} \uparrow e \\ g \longrightarrow * \\ \partial(e')^{-1}g' \\ \curvearrowleft \\ \boxed{\times} \uparrow e' \\ g' \end{array} * = * \begin{array}{c} \partial(e)^{-1}g \\ \curvearrowright \\ \boxed{\times} \uparrow e \\ g \longrightarrow * \\ \partial(e')^{-1}g' \\ \curvearrowleft \\ \boxed{\times} \uparrow e' \\ g' \end{array} * , \forall g, g' \in G, \forall e, e' \in E.$$

(23)

This interchange law is a well known consequence of the second Peiffer law $\partial(e) \triangleright e' = ee'e^{-1}, \forall e, e' \in E$, for crossed modules of groups. Therefore we can define the horizontal composition of two 2-morphisms as being:

$$* \begin{array}{c} \xrightarrow{\partial(e)^{-1}g} \\ \boxed{\times} \uparrow e \\ \xrightarrow{g} \end{array} * \begin{array}{c} \xrightarrow{\partial(e')^{-1}g'} \\ \boxed{\times} \uparrow e' \\ \xrightarrow{g'} \end{array} * \quad (24)$$

$$= * \begin{array}{c} \xrightarrow{\partial(e)^{-1}g} \\ \boxed{\times} \uparrow e \\ \xrightarrow{g} \end{array} * \begin{array}{c} \xrightarrow{\partial(e')^{-1}g'} \\ \boxed{\times} \uparrow e' \\ \xrightarrow{g'} \end{array} * = * \begin{array}{c} \xrightarrow{\partial(e)^{-1}g \partial(e')^{-1}g'} \\ \boxed{\times} \uparrow (g \triangleright e') e \\ \xrightarrow{gg'} \end{array} * \quad (25)$$

$$= * \begin{array}{c} \xrightarrow{\partial(e)^{-1}g} \\ \boxed{\times} \uparrow e \\ \xrightarrow{g} \end{array} * \begin{array}{c} \xrightarrow{\partial(e')^{-1}g'} \\ \boxed{\times} \uparrow e' \\ \xrightarrow{g'} \end{array} * = * \begin{array}{c} \xrightarrow{\partial(e)^{-1}g \partial(e')^{-1}g'} \\ \boxed{\times} \uparrow e (\partial(e)^{-1}g) \triangleright e' \\ \xrightarrow{gg'} \end{array} * \quad (26)$$

Remark 13. Let $\hat{\mathcal{X}} = (\partial: E \rightarrow G, \triangleright)$ be a pre-crossed module of groups. We can define a sesqui-category $\hat{\mathcal{C}}^\times(\hat{\mathcal{X}})$ of out $\hat{\mathcal{X}}$; see [Str96] for the definition of a sesqui-category, which is similar to the definition of a 2-category, except that the interchange law does not necessarily hold. The objects, morphisms and 2-morphisms of $\hat{\mathcal{C}}^\times(\hat{\mathcal{X}})$ are constructed exactly in the same way as before, and so are their vertical compositions and whiskerings. However, the interchange law does not hold in general, and therefore the horizontal composition of 2-morphisms is not well defined, given that the two sides of (23) may yield different results.

2.3.2 The bare algebras crossed module case

Let $\mathcal{A} = (\partial: A \rightarrow B, \triangleright, \triangleleft)$ be a crossed module of bare algebras. Consider the induced algebra map $\partial: A[[h]] \rightarrow B[[h]]$. We define a 2-groupoid $\mathcal{C}^+(\mathcal{A})$ out of \mathcal{A} . The set of 1-morphisms of $\mathcal{C}^+(\mathcal{A})$ is given by all arrows $* \xrightarrow{b} *$, with $b \in B[[h]]$. The compositions is:

$$* \xrightarrow{b} * \xrightarrow{b'} * = * \xrightarrow{bb'} *, \text{ where } b, b' \in B[[h]].$$

The set of 2-morphisms of $\mathcal{C}^+(\mathcal{A})$ is given by all 2-arrows of the form:

$$* \begin{array}{c} \xrightarrow{b+\partial(a)} \\ \boxed{+} \uparrow a \\ \xrightarrow{b} \end{array} * , \text{ where } a \in A[[h]] \text{ and } b \in B[[h]], \quad (27)$$

each of these having a vertical inverse of the form:

$$* \begin{array}{c} \xrightarrow{b} \\ \boxed{+} \uparrow (-a) \\ \xrightarrow{b+\partial(a)} \end{array} * .$$

If both b and $b + \partial(a)$ are invertible, we also have an horizontal inverse, of the form:

$$* \begin{array}{c} \xrightarrow{(b+\partial(a))^{-1}} \\ \boxed{+} \uparrow -(b+\partial(a))^{-1} \triangleright a \triangleleft b^{-1} \\ \xrightarrow{b^{-1}} \end{array} *$$

The vertical composition of 2-morphisms is:

$$\begin{array}{c}
 \begin{array}{ccc}
 & b+\partial(a)+\partial(a') & \\
 & \boxed{+} \uparrow a' & \\
 * & \xrightarrow{b+\partial(a)} & * \\
 & \boxed{+} \uparrow a & \\
 & b &
 \end{array}
 & = &
 \begin{array}{ccc}
 & b+\partial(a)+\partial(a') & \\
 & \boxed{+} \uparrow (a+a') & \\
 * & \xrightarrow{b} & * \\
 & b &
 \end{array}
 \end{array}
 , \text{ where } b, b' \in B[[h]] \text{ and } a, a' \in A[[h]].$$

The left and right whiskerings have the form:

$$\begin{array}{c}
 * \xrightarrow{b} * \begin{array}{ccc} & b'+\partial(a) & \\ & \boxed{+} \uparrow a & \\ & b' & \end{array} * = * \begin{array}{ccc} & bb'+b\partial(a) & \\ & \boxed{+} \uparrow b > a & \\ & bb' & \end{array} * , \text{ where } b, b' \in B[[h]] \text{ and } a \in A[[h]], \\
 * \begin{array}{ccc} & b+\partial(a) & \\ & \boxed{+} \uparrow a & \\ & b & \end{array} * \xrightarrow{b'} * = * \begin{array}{ccc} & bb'+\partial(a)b' & \\ & \boxed{+} \uparrow a < b' & \\ & bb' & \end{array} * , \text{ where } b, b' \in B[[h]] \text{ and } a \in A[[h]].
 \end{array}$$

The interchange law follows from equation (18), which holds in any crossed module of bare algebras. We can therefore define the horizontal composition of two 2-morphisms, as being:

$$\begin{array}{ccc}
 * \begin{array}{ccc} & \partial(a)+b & \\ & \boxed{+} \uparrow a & \\ & b & \end{array} * \begin{array}{ccc} & \partial(a')+b' & \\ & \boxed{+} \uparrow a' & \\ & b' & \end{array} * & = & * \begin{array}{ccc} & b+\partial(b > a' + a < b' + aa') & \\ & \boxed{+} \uparrow b > a' + a < b' + aa' & \\ & bb' & \end{array} * \quad (28)
 \end{array}$$

Remark 14. Similarly to Remark 13, if $\hat{A} = (\partial: A \rightarrow B, >, <)$ is a pre-crossed module of bare algebras, the construction just outlined does not define a 2-groupoid. We can still define 0, 1 and 2-cells and whiskerings, but the interchange law does not hold, in general, and therefore we can only build a sesqui-category $\hat{C}^+(\hat{A})$.

This result will be essential for relating the exact and fuzzy formulations of two-dimensional holonomy.

Theorem 15. Let $\mathcal{A} = (\partial: A \rightarrow B, >, <)$ be a unital crossed module of bare algebras. Consider the group crossed module $\mathcal{A}_\bullet^* = (\partial: A_\bullet^* \rightarrow B^*, \triangleright)$, defined in 2.2.4. There exists a 2-functor $\mathcal{T}_\mathcal{A}: \mathcal{C}^\times(\mathcal{A}_\bullet^*) \rightarrow \mathcal{C}^+(A)$, defining a natural transformations of functors $\mathcal{T}: \mathcal{C}^\times \Rightarrow \mathcal{C}^+$. It has the form:

$$\mathcal{T}_\mathcal{A} \left(* \xrightarrow{b} * \right) = * \xrightarrow{b} * , \text{ on 1-cells}$$

and, on 2-cells:

$$\mathcal{T}_\mathcal{A} \left(* \begin{array}{ccc} & \partial(e)^{-1}g & \\ & \boxed{\times} \uparrow e & \\ & g & \end{array} * \right) = * \begin{array}{ccc} & \partial(e)^{-1}g & \\ & \boxed{+} \uparrow (e^{-1} - 1) \triangleleft g & \\ & g & \end{array} * . \quad (29)$$

Remark 16. Note that if $e^{-1} = 1 + \sum_{n=1}^{+\infty} e_n h^n$, where $e_n \in A, \forall n \in \mathbb{N}$, and where 1 denotes the identity adjoined to $A[[h]]$, then $e^{-1} - 1 = \sum_{n=1}^{+\infty} e_n h^n$.

Proof: The sets of 1-morphisms of $\mathcal{C}^\times(\mathcal{A}_\bullet^*)$ and $\mathcal{C}^+(A)$ coincide. That vertical compositions are preserved by $\mathcal{T}_\mathcal{A}$ is consequence of the fact (for each $e, e' \in A_\bullet[[h]]$ and $g \in B[[h]]$, and putting $f = e^{-1} - 1, f' = e'^{-1} - 1$):

$$\begin{aligned}
 ((e'e')^{-1} - 1) \triangleleft g &= (e'^{-1}e^{-1} - 1) \triangleleft g = (f'f + f' + f) \triangleleft g = f \triangleleft g + f'f \triangleleft g + f' \triangleleft g \\
 &= f \triangleleft g + f' \triangleleft (\partial(f)g) + f' \triangleleft g = f \triangleleft g + f' \triangleleft (\partial(f) + 1_B)g \\
 &= (e^{-1} - 1) \triangleleft g + (e' - 1) \triangleleft (\partial(e)^{-1}g).
 \end{aligned}$$

(Note that the second Peiffer condition is explicitly used, and therefore this is a property exclusive to crossed modules of bare algebras, as opposed to pre-crossed modules.) The preservation of left and right whiskerings (respectively) follows from the fact (for $g, h \in B[[h]]$ and $e \in A_\bullet^*$ and putting $f = e^{-1} - 1$):

$$\begin{aligned} ((h \triangleright e^{-1}) - 1) \triangleleft (hg) &\doteq (h \triangleright f \triangleleft h^{-1}) \triangleleft (hg) = h \triangleright (f \triangleleft g) = h \triangleright ((e^{-1} - 1) \triangleleft g) \\ (e^{-1} - 1) \triangleleft (gh) &= f \triangleleft (gh) = (f \triangleleft g) \triangleleft h = ((e^{-1} - 1) \triangleleft g) \triangleleft h. \end{aligned}$$

□

2.4 Crossed modules of Hopf algebras

2.4.1 Hopf algebraic conventions.

Recall some Hopf algebra conventions; see [Kas95, Maj95]. Let $H = (H, \Delta, \epsilon)$ be a bialgebra, with identity $1 = 1_H$ and counit $\epsilon: H \rightarrow \mathbb{C}$. The coproduct in H will be written in Sweedler's notation, as in [Kas95]:

$$\Delta(x) = \sum_{(x)} x' \otimes x'', \text{ for each } x \in H.$$

Suppose that I is a bialgebra, with identity 1_I . We say that I is a H -module algebra if there exists a left-action $\rho: H \times I \rightarrow I$, explicitly $\rho: (x, v) \in H \times I \mapsto x \triangleright_\rho v \in I$, such that:

- $1_H \triangleright_\rho v = v$ for each $v \in I$.
- $x \triangleright_\rho (uv) = \sum_{(x)} (x' \triangleright_\rho u) (x'' \triangleright_\rho v)$, for each $u, v \in I$, and each $x \in H$.
- $x \triangleright_\rho 1_I = \epsilon(x)1_I$, for each $x \in H$.

With the algebra action $\rho: H \otimes I \rightarrow I$, I is said to be an H -module coalgebra if we have:

- $\Delta(x \triangleright_\rho v) = \sum_{(x)} \sum_{(v)} (x' \triangleright_\rho v') \otimes (x'' \triangleright_\rho v'')$ for each $x \in H$ and each $v \in I$.
- $\epsilon(x \triangleright_\rho v) = \epsilon(x)\epsilon(v)$ for each $x \in H$ and each $v \in I$.

The following is well known and a proof appears in [Maj95].

Theorem 17 (Majid). *If I is an H -module algebra and coalgebra, under the left action ρ , then we can define an algebra $I \otimes_\rho H$, with underlying vector space $I \otimes H$, with identity $1_I \otimes 1_H$, and with product:*

$$(u \otimes x)(v \otimes y) = \sum_{(x)} (u x' \triangleright_\rho v) \otimes (x''y), \text{ where } u, v \in I \text{ and } x, y \in H.$$

If, additionally, the following compatibility condition holds:

$$\sum_{(x)} x' \otimes (x'' \triangleright_\rho v) = \sum_{(x)} x'' \otimes (x' \triangleright_\rho v), \text{ for each } x \in H \text{ and } v \in I, \quad (30)$$

then we have a bialgebra structure in $I \otimes_\rho H$, with coproduct:

$$\Delta(u \otimes x) = \sum_{(u)} \sum_{(x)} (u' \otimes x') \otimes (u'' \otimes x''), \text{ where } x \in H \text{ and } u \in I,$$

(the compatibility condition (30) is only used for proving that the coproduct is an algebra morphism), and:

$$\epsilon(u \otimes x) = \epsilon(u)\epsilon(x), \text{ where } x \in H \text{ and } u \in I.$$

If, in addition, H and I are Hopf algebras, then $I \otimes_\rho H$ is a Hopf algebra, with antipode:

$$S(u \otimes x) = (1_I \otimes S(x)) (S(v) \otimes 1_H), \text{ where } x \in H \text{ and } u \in I.$$

Moreover the inclusions of H and I in $I \otimes_\rho H$, namely:

$$i_H: x \in H \mapsto 1_I \otimes x \in I \otimes_\rho H, \quad i_I: v \in I \mapsto v \otimes 1_I \in I \otimes_\rho H$$

are bialgebra morphisms (and Hopf algebra morphisms if H and I are Hopf algebras).

2.4.2 The adjoint representation. Primitive and group like elements

All of this is well known. If H is a Hopf algebra, then H has a H -module algebra structure, given by:

$$\rho(x \otimes y) = \sum_{(x)} x' y S(x'') \doteq x \triangleright_{\text{ad}} y,$$

called the adjoint representation. (In general we do not have an H -module coalgebra structure, unless, for example, H is cocommutative. In fact $\Delta(x \triangleright y) = \sum_{(x)} \sum_{(y)} x' y' S(x''') \otimes x'' y'' S(x''')$, for each $x, y \in H$.)

Recall that an element $x \in H$ is said to be:

- Grouplike, if $\Delta(x) = x \otimes x$.
- Primitive, if $\Delta(x) = x \otimes 1 + 1 \otimes x$.

It is easy to see that $\epsilon(x) = 1$, if x is group-like, and $\epsilon(x) = 0$, if x is primitive. Any group-like element x is invertible, being $S(x) = x^{-1}$. If x is primitive then $S(x) = -x$. Therefore:

Lemma 18 (The adjoint action of group-like and of primitive elements). *Let $y \in H$. Then:*

- If $g \in H$ is group-like then $g \triangleright_{\text{ad}} y = gyg^{-1}$.
- If $x \in H$ is primitive then $x \triangleright_{\text{ad}} y = xy - yx$.

2.4.3 Crossed modules and pre-crossed modules of Hopf algebras

The following definition appeared in [Maj12]. It is more restrictive than the one appearing in [FW11, FLV07] in that condition (30) is imposed from the beginning.

Definition 19 (Crossed modules and pre-crossed module of Hopf algebras). *A Hopf algebra crossed module $\mathcal{H} = (\partial: I \rightarrow H, \rho)$ is given by:*

- A map $\partial: I \rightarrow H$ of Hopf algebras.
- A left action $\rho: x \otimes v \in H \otimes I \mapsto x \triangleright_{\rho} v \in I$, which turns I into an H -module algebra and coalgebra, satisfying moreover the compatibility condition (30).

We furthermore impose the first and second Peiffer laws, as displayed below:

1. $\partial(x \triangleright_{\rho} v) = x \triangleright_{\text{ad}} \partial(v)$ (first Peiffer Law), where $x \in H$ and $v \in I$.
2. $\partial(u) \triangleright_{\rho} v = u \triangleright_{\text{ad}} v$ (second Peiffer Law), where $u, v \in I$.

If the first Peiffer law is satisfied (but not necessarily the second) for $\hat{\mathcal{H}} = (\partial: I \rightarrow H, \rho)$, then $\hat{\mathcal{H}}$ is called a pre-crossed module, of Hopf algebras.

The following theorem appeared in [Maj95], being a simple extension of Theorem 17

Theorem 20 (Majid). *Suppose that $\mathcal{H} = (\partial: I \rightarrow H, \rho)$ is a pre-crossed module of Hopf algebras. We have Hopf algebra maps (called source and target maps) $s, t: I \otimes_{\rho} H \rightarrow H$, which read on generators:*

$$s: v \otimes x \in I \otimes_{\rho} H \mapsto \epsilon(v)x \in H, \quad t: v \otimes x \in I \otimes_{\rho} H \mapsto \partial(v)x \in H.$$

Moreover, together with the inclusion $i_H: H \rightarrow I \otimes_{\rho} H$ (called identity map) these define a graph in the category of Hopf algebras, namely we have $s \circ i_H = \text{id}_H$ and $t \circ i_H = \text{id}_H$.

Remark 21. *Even though we do not assume our Hopf algebras to be cocommutative, the conditions we impose are still quite restrictive, however perfectly adapted to the type of examples we have in mind: crossed modules of Hopf algebras arising from differential crossed modules. In [Maj95] a much more general definition of crossed modules of (braided) Hopf algebras appears, enabling for genuinely non-cocommutative examples.*

2.4.4 From crossed module of Hopf algebras to group crossed modules

Let H be a Hopf algebra. We let $H[[h]]$ be the ring of formal power series over H . This inherits an obvious algebra structure, with ground ring $\mathbb{C}[[h]]$. Namely $\left(\sum_{n=0}^{+\infty} h^n a_n\right) \left(\sum_{m=0}^{+\infty} h^m b_m\right) = \sum_{n=0}^{+\infty} h^n \sum_{i+j=n} a_i b_j$, clearly $\mathbb{C}[[h]]$ -bilinear. Note, however, that $H[[h]]$ is not necessarily a bialgebra, with ground ring $\mathbb{C}[[h]]$. The (very well known) point here is that, given a formal power series $\sum_{n=0}^{+\infty} h^n a_n \in H[[h]]$, its coproduct would be $\sum_{n=0}^{+\infty} h^n \Delta(a_n)$, which lives in $(H \otimes H)[[h]]$, strictly containing $H[[h]] \otimes_{\mathbb{C}[[h]]} H[[h]]$. Nevertheless, the $\mathbb{C}[[h]]$ -algebra $H[[h]]$ can be seen as being a Hopf algebra, however such that the coproduct Δ is a map $\Delta: H[[h]] \rightarrow (H \otimes H)[[h]] \supseteq H[[h]] \otimes_{\mathbb{C}[[h]]} H[[h]]$. (This can be dealt with fully precisely by considering a completion of the tensor product of $\mathbb{C}[[h]]$ -modules, as explained, for example, in [Kas95, XVI.1]. We will not use that language since a completely algebraic proof can always be given.)

Definition 22 (Group-like element in $H[[h]]$). *Let H be a Hopf algebra. An element of $x \in H[[h]]$ is said to be group like if $\Delta(x) = x \otimes x$. Explicitly, $x = \sum_{n=0}^{+\infty} h^n x_n$ is said to be group like if $\sum_{n=0}^{+\infty} h^n \Delta(x_n) = \sum_{n=0}^{+\infty} h^n \sum_{i+j=n} x_i \otimes x_j$, that is, using Sweedler's notation:*

$$\Delta(x) = x \otimes x \iff \sum_{n=0}^{+\infty} h^n \sum_{(x_n)} x'_n \otimes x''_n = \sum_{n=0}^{+\infty} h^n \sum_{i+j=n} x_i \otimes x_j.$$

For example the identity 1_H of H (and therefore of $H[[h]]$) is group like. Group-like elements of $H[[h]]$ are closed under multiplication. Let $a \in H$ be a primitive element. Then $\exp(ha) = \sum_{n=0}^{+\infty} h^n \frac{a^n}{n!}$ is a group like element of $H[[h]]$. If $x = \sum_{n=0}^{+\infty} x_n h^n \in H[[h]]$ is group like, then, similarly to the case of Hopf algebras over a field, $\epsilon(x) = \sum_{n=0}^{+\infty} h^n \epsilon(x_n) = 1_H$ and $xS(x) = 1_H = S(x)x$. The antipode S is extended $\mathbb{C}[[h]]$ -linearly.

Let $\mathcal{H} = (\partial: I \rightarrow H, \rho)$ be a crossed module of Hopf algebras. Extend the action of H on I , $\mathbb{C}[[h]]$ -linearly:

$$\left(\sum_{n=0}^{+\infty} x_n h^n\right) \triangleright_{\rho} \left(\sum_{m=0}^{+\infty} v_m h^m\right) = \sum_{n=0}^{+\infty} h^n \sum_{i+j=n} x_i \triangleright_{\rho} v_j, \text{ where } x_n \in H \text{ and } v_n \in I, \text{ for each } n \in \mathbb{N}.$$

Let H_{gl}^* and I_{gl}^* be the groups of invertible group like elements of $H[[h]]$ and of $I[[h]]$, whose underlying power series start at the identity (of H or I). Consider the induced map $\partial: I[[h]] \rightarrow H[[h]]$, sending group like elements to group-like elements, and invertible elements to invertible elements. The restriction $\partial: I_{\text{gl}}^* \rightarrow H_{\text{gl}}^*$ of $\partial: I[[h]] \rightarrow H[[h]]$ is a map of groups. In addition:

Lemma 23. *The action \triangleright_{ρ} of $H[[h]]$ on $I[[h]]$ restricts to a group action of H_{gl}^* on I_{gl}^* , by automorphisms.*

Proof: Let $x, y \in H_{\text{gl}}^*$ and $u, v \in I_{\text{gl}}^*$. Put $x = \sum_{n=0}^{+\infty} h^n x_n$, $v = \sum_{n=0}^{+\infty} h^n v_n$, $u = \sum_{n=0}^{+\infty} h^n u_n$. That $(xy) \triangleright_{\rho} u = x \triangleright_{\rho} (y \triangleright_{\rho} u)$ follows since ρ is an algebra action. That $x \triangleright_{\rho} (uv) = (x \triangleright_{\rho} u)(x \triangleright_{\rho} v)$ follows from:

$$\begin{aligned} x \triangleright_{\rho} (uv) &= \sum_{n=0}^{+\infty} h^n \sum_{i+j+k=n} x_i \triangleright_{\rho} (u_j v_k) = \sum_{n=0}^{+\infty} h^n \sum_{i+j+k=n} \sum_{(x_i)} (x'_i \triangleright_{\rho} u_j) (x''_i \triangleright_{\rho} v_k) \\ &= \sum_{n=0}^{+\infty} h^n \sum_{i+j+k=n} \sum_{o+p=i} (x_o \triangleright_{\rho} u_j) (x_p \triangleright_{\rho} v_k) = \sum_{n=0}^{+\infty} h^n \sum_{o+p+j+k=n} (x_o \triangleright_{\rho} u_j) (x_p \triangleright_{\rho} v_k) = (x \triangleright_{\rho} u) (x \triangleright_{\rho} v). \end{aligned}$$

In fact what we proved is that the action of H_{gl}^* on $I[[h]]$ is a group action by algebra isomorphisms. \square

Lemma 24. *Let $x, y \in A[[h]]$, where A is a Hopf algebra, with x invertible and group-like. Then:*

$$x \triangleright_{\text{ad}} y = x y x^{-1}. \quad (31)$$

Proof: Put $x = \sum_{n=0}^{+\infty} h^n x_n$ and $y = \sum_{n=0}^{+\infty} h^n y_n$. We have:

$$\begin{aligned} x \triangleright_{\text{ad}} y &= \left(\sum_{n=0}^{+\infty} h^n x_n\right) \triangleright_{\text{ad}} \left(\sum_{n=0}^{+\infty} h^n y_n\right) = \sum_{n=0}^{+\infty} h^n \sum_{i+j=n} x_i \triangleright_{\text{ad}} y_j = \sum_{n=0}^{+\infty} h^n \sum_{i+j=n} \sum_{(x_i)} x'_i y_j S(x''_i) \\ &= \sum_{n=0}^{+\infty} h^n \sum_{i+j=n} \sum_{o+p=i} x_o y_j S(x_p) = \sum_{n=0}^{+\infty} h^n \sum_{o+p+j=n} x_o y_j S(x_p) = xyx^{-1}. \end{aligned}$$

□

Combining the two previous lemmas we see that.

Theorem 25. *Let $\mathcal{H} = (\partial: I \rightarrow H, \rho)$ be a crossed module of Hopf algebras. We have a crossed module of groups $\mathcal{H}_{\mathfrak{gl}}^* = (\partial: I_{\mathfrak{gl}}^* \rightarrow H_{\mathfrak{gl}}^*, \triangleright_{\rho})$.*

Remark 26. *Consider a Hopf algebra crossed module $\mathcal{H} = (\partial: I \rightarrow H, \rho)$. Looking at the two previous lemmas, given $x \in H_{\mathfrak{gl}}^*$, then, for all $y \in I[[h]]$, we have $\partial(x \triangleright y) = x \triangleright_{\text{ad}} \partial(y) = x \partial(y) x^{-1}$. Define $\hat{I}_{\mathfrak{gl}}^* \supset I_{\mathfrak{gl}}^*$ as being the group of invertible elements of $I[[h]]$, mapping, through $\partial: I[[h]] \rightarrow H[[h]]$, to group-like elements of $H[[h]]$, of the form $1_H + \sum_{n=1}^{+\infty} h^n x_n$. Clearly $\hat{\mathcal{H}}_{\mathfrak{gl}}^* = (\partial: \hat{I}_{\mathfrak{gl}}^* \rightarrow H_{\mathfrak{gl}}^*, \triangleright_{\rho})$ is a pre-crossed module of groups.*

2.4.5 From crossed modules of Hopf algebras to differential (pre)-crossed modules

Let H be an Hopf algebra. Let $g, x, y \in H$. If g is group-like then $g \triangleright_{\text{ad}} y = gyg^{-1}$. If x is primitive then $x \triangleright_{\text{ad}} y = [x, y] = xy - yx$. The vector space of primitive elements of H is a Lie algebra, denoted by $\text{Prim}(H)$, and the adjoint action by a group-like element $g \in H$ is a Lie algebra map $x \in \text{Prim}(H) \mapsto g \triangleright_{\text{ad}} x \in \text{Prim}(H)$. We denote by $\widehat{\text{Prim}}(I)$ the vector space of elements of I mapping through ∂ to $\text{Prim}(H)$.

Proposition 27. *Let $\mathcal{H} = (\partial: I \rightarrow H, \rho)$ be a Hopf algebra crossed module. Then we have a differential crossed module $\text{Prim}(\mathcal{H}) = (\partial: \text{Prim}(I) \rightarrow \text{Prim}(H), \triangleright_{\rho})$ and a differential pre-crossed module $\widehat{\text{Prim}}(\mathcal{H}) = (\partial: \widehat{\text{Prim}}(I) \rightarrow \text{Prim}(H), \triangleright_{\rho})$*

We will consider the following bilinear map in $\widehat{\text{Prim}}(I)$, vanishing in $\text{Prim}(I) \otimes \text{Prim}(I)$:

$$u \otimes v \mapsto \langle u, v \rangle \doteq [u, v] - \partial(u) \triangleright v, \text{ where } u, v \in \widehat{\text{Prim}}(I). \quad (32)$$

This map measures how far the differential pre-crossed module $\widehat{\text{Prim}}(\mathcal{H})$ is from being a crossed module. It will be used when discussing the relation between holonomy and curvature; see Subsection 4.2.2.

2.5 Universal enveloping algebras

2.5.1 Review on universal enveloping algebras

Consider the underlying vector space functor, from the category of (unital) associative algebras to the category of vector spaces, and let \mathcal{T} be its left adjoint, the tensor algebra functor, thus:

$$\mathcal{T}(V) = \bigoplus_{n=0}^{+\infty} V^{n\otimes}, \text{ where } V^{n\otimes} = \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}}, \text{ and } V^0 \doteq \mathbb{C},$$

with multiplication being the tensor product. Given a vector space V , the tensor algebra $\mathcal{T}(V)$ can be given a Hopf algebra structure by putting $\Delta(x) = x \otimes 1 + 1 \otimes x$ and $\epsilon(x) = 0$, on generators $x \in V$, with antipode $S(x) = -x, \forall x \in V$. If B is a basis of V , then $\mathcal{T}(V)$ is isomorphic to the algebra $\mathbb{C}\langle B \rangle$ of non commutative polynomials, with coefficients in \mathbb{C} , and with a formal variable for each element of B . Therefore $\mathcal{T}(\mathbb{C}) \cong \mathbb{C}[x]$.

We follow [Var84], closely. If A is a bare algebra, recall that an algebra derivation $f: A \rightarrow A$ is a linear map such that $f(aa') = f(a)a' + af(a'), \forall a, a' \in A$. Given a Lie algebra \mathfrak{g} , a Lie algebra derivation is a map $f: \mathfrak{g} \rightarrow \mathfrak{g}$, such that $f([x, y]) = [f(x), y] + [x, f(y)], \forall x, y \in \mathfrak{g}$. We denote by $\text{Der}(A)$ and $\text{Der}(\mathfrak{g})$ the Lie algebras of algebra derivations of A and of Lie algebra derivations of \mathfrak{g} . Any vector space map $f: V \rightarrow V$ gives rise to a unique derivation $\text{der}(f)$ of $\mathcal{T}(V)$, extending f . This yields a Lie algebra map $\text{der}: f \in \mathfrak{gl}(V) \mapsto \text{der}(f) \in \text{Der}(\mathcal{T}(V))$, where $\mathfrak{gl}(V)$ is the Lie algebra of linear maps $V \rightarrow V$.

Let \mathfrak{g} be a Lie algebra. We denote its universal enveloping algebra by $\mathcal{U}(\mathfrak{g})$. This is the quotient of $\mathcal{T}(\mathfrak{g})$ by the bilateral ideal L generated by $\{XY - YX - [X, Y], \forall X, Y \in \mathfrak{g}\} \subset \mathcal{T}(\mathfrak{g})$. This is a Hopf-algebra ideal: $\Delta(L) \subset L \otimes \mathcal{T}(\mathfrak{g}) + \mathcal{T}(\mathfrak{g}) \otimes L$, $\epsilon(L) = \{0\}$ and $S(L) \subset L$. Therefore the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is a Hopf algebra. Given $X \in \mathfrak{g}$ the coproduct is $\Delta(X) = X \otimes 1 + 1 \otimes X$, and the counit is $\epsilon(X) = 0$. If $f: \mathfrak{g} \rightarrow \mathfrak{g}$ is a Lie algebra derivation, then $\text{der}(f)(L) \subset L$, thus we have a Lie algebra map $\text{der}: \text{Der}(\mathfrak{g}) \rightarrow \text{Der}(\mathcal{U}(\mathfrak{g}))$.

Recall that the structure of $\mathcal{U}(\mathfrak{g})$ is unraveled by the Poincaré-Birkhoff-Witt (PBW) theorem. One of the numerous equivalent ways to state it is the following: if $\{X_i, i \in I\}$ is a basis of \mathfrak{g} , where I is a totally ordered set, then $\{1\} \cup \{X_{i_1} X_{i_2} \cdots X_{i_n}, n \in \mathbb{N}, i_1 \leq i_2 \leq \cdots \leq i_n\}$ is a basis of $\mathcal{U}(\mathfrak{g})$. The PBW theorem allows us to consider \mathfrak{g} as being, canonically a subvector space of \mathfrak{g} . (A fact that will be used without explanation.)

2.5.2 The restricted universal enveloping algebra $\mathcal{U}^0(\mathfrak{g})$

Given a Lie algebra \mathfrak{g} , we will also consider the restricted universal enveloping algebra $\mathcal{U}^0(\mathfrak{g})$, an algebra without unit, which is defined as being the kernel $\ker(\epsilon) \subset \mathcal{U}(\mathfrak{g})$ of the counit $\epsilon: \mathcal{U}(\mathfrak{g}) \rightarrow \mathbb{C}$, therefore being the augmentation ideal of $\mathcal{U}(\mathfrak{g})$. By the PBW theorem, $\{X_{i_1} X_{i_2} \dots X_{i_n}, n \in \mathbb{N}, i_1 \leq i_2 \leq \dots \leq i_n\}$ is a basis of $\mathcal{U}^0(\mathfrak{g})$. Clearly (from the PBW theorem) $\mathcal{U}^0(\mathfrak{g})$ is the smallest non-unital subalgebra of $\mathcal{U}(\mathfrak{g})$ containing \mathfrak{g} . One recovers $\mathcal{U}(\mathfrak{g})$ from $\mathcal{U}^0(\mathfrak{g})$ by formally adjoining an identity; in other words $\mathcal{U}^0(\mathfrak{g})_{\bullet} \cong \mathcal{U}(\mathfrak{g})$.

2.5.3 The sharp $\xi_A^{\sharp}: \mathcal{U}(\text{Lie}(A)) \rightarrow A_{\bullet}$ and flat $\xi_A^{\flat}: \mathcal{U}^0(\text{Lie}(A)) \rightarrow A$ counits for a bare algebra A

Recall the adjunction between unital algebras and Lie algebras, whose right adjoint associates to a unital algebra A its commutator algebra $\text{Lie}(A)$. The left adjoint is precisely the universal enveloping algebra functor. The counit of this adjunction is the natural transformation ξ , which, to a unital algebra A , associates the unital algebra map $\xi_A: \mathcal{U}(\text{Lie}(A)) \rightarrow A$, which, on generators $x \in A$, is $x \mapsto x$. Given a Lie algebra \mathfrak{g} and a unital algebra A , any Lie algebra map $f: \mathfrak{g} \rightarrow \text{Lie}(A)$ yields a unique unital algebra map $\hat{f}: \mathcal{U}(\mathfrak{g}) \rightarrow A$, which, on generators $x \in \mathfrak{g}$, is $x \mapsto f(x)$. Let us now suppose that A does not necessarily have an identity.

Definition 28 (Sharp counit). *Let A be a bare algebra. Let $A_{\bullet} = \mathbb{C} \oplus A$ be the unital algebra obtained from A by adjoining a unit. Let $\iota: A \rightarrow A_{\bullet}$ be the inclusion map $x \mapsto (0, x)$, inducing a Lie algebra map $\iota: \text{Lie}(A) \rightarrow \text{Lie}(A_{\bullet})$. The sharp counit is the unital algebra map:*

$$\xi_A^{\sharp}: \mathcal{U}(\text{Lie}(A)) \rightarrow A_{\bullet},$$

universally provided by the Lie algebra map $\iota: \text{Lie}(A) \rightarrow \text{Lie}(A_{\bullet})$. Therefore, on generators $x \in A$, we have $\xi_A^{\sharp}(x) = (0, x)$, and finally $\xi_A^{\sharp}(1) = (1, 0)$. Therefore, clearly $\xi_A^{\sharp}(\mathcal{U}^0(\text{Lie}(A))) = \{0\} \times A \subsetneq A_{\bullet}$.

Let A be a bare algebra. Consider the obvious algebra isomorphism $p_A: \{0\} \times A \subset A_{\bullet} \rightarrow A$.

Definition 29 (Flat counit). *Let A be a bare algebra. The flat counit is the bare algebra map:*

$$\xi_A^{\flat}: \mathcal{U}^0(\text{Lie}(A)) \rightarrow A,$$

given by $\xi_A^{\flat} = p_A \circ \epsilon_A^{\sharp}$. Therefore, on generators $a \in A$, we have: $\xi_A^{\flat}(a) = a$. Note that ξ_A^{\flat} is surjective.

The categorical explanation of the flat counit is the following: we have an adjunction between the category of Lie algebras and the category of (not-necessarily unital) algebras; i.e. bare algebras. The right adjoint assigns to any associative algebra its commutator Lie algebra, whereas the left adjoint is the restricted universal enveloping algebra. The flat counit is exactly the counit of this adjunction.

2.5.4 From differential crossed modules to crossed module of Hopf algebras

As proved in [Maj12], a differential crossed module $\mathfrak{X} = (\partial: \mathfrak{e} \rightarrow \mathfrak{g}, \triangleright)$ gives rise to a crossed module of Hopf algebras $\mathcal{U}(\mathfrak{X}) = (\partial: \mathcal{U}(\mathfrak{e}) \rightarrow \mathcal{U}(\mathfrak{g}), \rho)$. Let us spell out the construction. The algebra map $\partial: \mathcal{U}(\mathfrak{e}) \rightarrow \mathcal{U}(\mathfrak{g})$ is the map functorially given by the Lie algebra map $\partial: \mathfrak{e} \rightarrow \mathfrak{g}$. The action \triangleright of \mathfrak{g} on \mathfrak{e} , by derivations, yields a Lie algebra map $\mathfrak{g} \rightarrow \text{Der}(\mathfrak{e})$, and therefore, composing with $\text{der}: \text{Der}(\mathfrak{e}) \rightarrow \text{Der}(\mathcal{U}(\mathfrak{e}))$, a Lie algebra map $\mathfrak{g} \rightarrow \text{Der}(\mathcal{U}(\mathfrak{e})) \subset \mathfrak{gl}(\mathcal{U}(\mathfrak{e}))$, the Lie algebra of linear maps $\mathcal{U}(\mathfrak{e}) \rightarrow \mathcal{U}(\mathfrak{e})$. Thence we have an algebra map $\mathcal{U}(\mathfrak{g}) \rightarrow \text{hom}(\mathcal{U}(\mathfrak{e}))$, where $\text{hom}(\mathcal{U}(\mathfrak{e}))$ is the algebra of linear maps $\mathcal{U}(\mathfrak{e}) \rightarrow \mathcal{U}(\mathfrak{e})$, inducing an action:

$$\rho: (x, v) \in \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{e}) \mapsto x \triangleright v \in \mathcal{U}(\mathfrak{e}).$$

Theorem 30 (Majid). *If $\mathfrak{X} = (\partial: \mathfrak{e} \rightarrow \mathfrak{g}, \triangleright)$ is a differential crossed module then $\mathcal{U}(\mathfrak{X}) = (\partial: \mathcal{U}(\mathfrak{e}) \rightarrow \mathcal{U}(\mathfrak{g}), \rho)$ is a crossed module of Hopf algebras.*

2.5.5 The crossed module counit

Suppose that $\mathcal{A} = (\partial: A \rightarrow B, \triangleright, \triangleleft)$ is a unital crossed module of bare algebras. Pass to the associated crossed module of Lie algebras $\text{Lie}(\mathcal{A}) = (\partial: \text{Lie}(A) \rightarrow \text{Lie}(B), \triangleright)$; see 2.2.3. This will yield a crossed module of Hopf algebras $\mathcal{U}(\text{Lie}(\mathcal{A})) = (\partial: \mathcal{U}(\text{Lie}(A)) \rightarrow \mathcal{U}(\text{Lie}(B)), \rho)$. Recall the notation of 2.2.4, 2.4.4 and 2.5.3. Consider the flat counit $\xi_A^{\flat}: \mathcal{U}^0(\text{Lie}(A)) \rightarrow A$ and the usual counit $\xi_B: \mathcal{U}(\text{Lie}(B)) \rightarrow B$. Let us look at

the pair (ξ_A^b, ξ_B) . We will denote by the same symbols the extensions $\xi_A^b: \mathcal{U}^0(\text{Lie}(A))[[h]] \rightarrow A[[h]]$ and $\xi_B: \mathcal{U}(\text{Lie}(B))[[h]] \rightarrow B[[h]]$. Clearly the diagram below (in the category of bare algebras) commutes:

$$\begin{array}{ccc} \mathcal{U}^0(\text{Lie}(A))[[h]] & \xrightarrow{\partial} & \mathcal{U}(\text{Lie}(B))[[h]] \\ \xi_A^b \downarrow & & \downarrow \xi_B \\ A[[h]] & \xrightarrow{\partial} & B[[h]] \end{array}$$

We cannot conclude that (ξ_A^b, ξ_B) is a map of crossed modules of bare algebras, among other reasons since $\mathcal{U}(\text{Lie}(A)) = (\partial: \mathcal{U}^0(\text{Lie}(A)) \rightarrow \mathcal{U}(\text{Lie}(B)), \triangleright_\rho)$ does not, a priori, have the structure of such; this will be expanded in 2.5.6. However, passing to the associated crossed modules of groups, the pair (ξ_A^b, ξ_B) induces a group crossed module map, the crossed module counit, $\text{Proj}_A: \mathcal{U}(\text{Lie}(A))_{\text{gl}}^* \rightarrow \mathcal{A}_\bullet^*$. We now address this.

Lemma 31. *For each $x \in B \subset \mathcal{U}(\text{Lie}(B))$, and each $a \in \mathcal{U}^0(\text{Lie}(A))$, we have:*

$$\xi_A^b(x \triangleright a) = \xi_B(x) \triangleright \xi_A^b(a) - \xi_A^b(a) \triangleleft \xi_B(x).$$

Proof: It suffices proving it for $a = v_1 * \cdots * v_n$, where $v_i \in A$. Use $*$ to denote the product in $\mathcal{U}(\text{Lie}(A))$.

$$\begin{aligned} \xi_A^b(x \triangleright a) &= \xi_A^b(x \triangleright (v_1 * v_2 * \cdots * v_n)) \\ &= \xi_A^b((x \triangleright v_1) * v_2 * \cdots * v_n) + \xi_A^b(v_1 * (x \triangleright v_2) * \cdots * v_n) + \cdots + \xi_A^b(v_1 * v_2 * \cdots * (x \triangleright v_n)) \\ &= (x \triangleright v_1)v_2 \dots v_n + v_1(x \triangleright v_2) \dots v_n + \cdots + v_1 v_2 \dots (x \triangleright v_n) \\ &= (x \triangleright v_1 - v_1 \triangleleft x)v_2 \dots v_n + v_1(x \triangleright v_2 - v_2 \triangleleft x) \dots v_n + \cdots + v_1 v_2 \dots (x \triangleright v_n - v_n \triangleleft x) \\ &= x \triangleright (v_1 v_2 \dots v_n) - (v_1 v_2 \dots v_n) \triangleleft x = \xi_B(x) \triangleright \xi_A^b(a). \end{aligned}$$

(Note that the left and right actions \triangleright and \triangleleft are fully interchangeable.) \square

We emphasize that, in general, if $w \in \mathcal{U}(\text{Lie}(B))$ then $\xi_A^b(w \triangleright a) \neq \xi_B(w) \triangleright \xi_A^b(a) - \xi_A^b(a) \triangleleft \xi_B(w)$. Consider the algebra $\text{hom}(A)$ of linear maps $A \rightarrow A$. Then we have a unital algebra map $R: \mathcal{U}(\text{Lie}(B)) \rightarrow \text{hom}(A)$, which, on algebra generators $x \in B$, is such that $R(x)(a) = x \triangleright a - a \triangleleft x = x \triangleright a$. (Considering the Lie algebra action \triangleright of $\text{Lie}(B)$ on A , this is just the universal enveloping algebra version of \triangleright .)

Lemma 32. *For each w in $\mathcal{U}(\text{Lie}(B))$, and each $a \in A$, we have:*

$$R(w)(a) = \sum_{(w)} \xi_B(w') \triangleright a \triangleleft \xi_B(S(w'')). \quad (33)$$

Proof: Let us see that the right-hand-side of (33) defines an algebra map $T: \mathcal{U}(\text{Lie}(B)) \rightarrow \text{hom}(A)$, where

$$T(w)(a) = \sum_{(w)} \xi_B(w') \triangleright a \triangleleft \xi_B(S(w'')), \text{ where } a \in A \text{ and } w \in \mathcal{U}(\text{Lie}(B)).$$

If $x, y \in \mathcal{U}(\text{Lie}(B))$ and $a \in A$ we have:

$$\begin{aligned} T(xy)(a) &\doteq \sum_{(xy)} \xi_B((xy)') \triangleright a \triangleleft \xi_B(S((xy)'')) = \sum_{(x)} \sum_{(y)} \xi_B(x'y') \triangleright a \triangleleft \xi_B(S(y'')S(x'')) \\ &= \sum_{(x)} \sum_{(y)} \xi_B(x') \triangleright (\xi_B(y') \triangleright a \triangleleft \xi_B(S(y''))) \triangleleft \xi_B(S(x'')) = T(x)(T(y)(a)). \end{aligned}$$

Now just use the fact that, clearly, R and T coincide on the algebra generators $x \in B$, all primitive. \square

Now consider the associated algebras over $\mathbb{C}[[h]]$, $\mathcal{U}(\text{Lie}(B))[[h]]$, $\mathcal{U}^0(\text{Lie}(A))[[h]]$, $B[[h]]$ and $A[[h]]$.

Lemma 33. *For each $w \in B \subset \mathcal{U}(\text{Lie}(B))_{\text{gl}}^*$, and each $a \in \mathcal{U}^0(\text{Lie}(A))[[h]]$, it holds that:*

$$\xi_A^b(w \triangleright a) = \xi_B(w) \triangleright \xi_A^b(a) \triangleleft \xi_B(w)^{-1}.$$

Proof: If w is group like, then $\Delta(w) = w \otimes w$, being $S(w) = w^{-1}$. In particular we have a bare algebra map $a \in \mathcal{U}^0(\text{Lie}(A))[[h]] \mapsto w \triangleright a \in \mathcal{U}^0(\text{Lie}(A))[[h]]$. Thus also $a \in \mathcal{U}^0(\text{Lie}(A))[[h]] \mapsto \xi_A^b(w \triangleright a) \in A[[h]]$ is a bare algebra map. Clearly $a \in \mathcal{U}^0(\text{Lie}(A))[[h]] \mapsto \xi_B(w) \triangleright \xi_A^b(a) \triangleleft \xi_B(w)^{-1} \in A[[h]]$ is a bare algebra map. The latter two maps coincide on generators $a \in A \subseteq \mathcal{U}^0(\text{Lie}(A))[[h]]$, by the previous lemma. \square

Theorem 34 (Crossed module counit). *Let $\mathcal{A} = (\partial: A \rightarrow B, \triangleright, \triangleleft)$ be a unital crossed module of bare algebras over \mathbb{C} . The algebra maps $\xi_A^\sharp: \mathcal{U}(\text{Lie}(A)) \rightarrow \mathcal{A}_\bullet$ and $\xi_B: \mathcal{U}(\text{Lie}(B)) \rightarrow B$ induce a group crossed module map $\text{Proj}_{\mathcal{A}}: \mathcal{U}(\text{Lie}(A))_{\text{gl}}^* \rightarrow \mathcal{A}_\bullet^*$, called the crossed module counit.*

Proof: We must prove that $\text{Proj}_{\mathcal{A}}$ intertwines the group actions. Let $1_{\mathcal{U}(\text{Lie}(A))} + \sum_{n=1}^{+\infty} h^n a_n \in \mathcal{U}(\text{Lie}(A))_{\text{gl}}^*$. Let $x \in \mathcal{U}(\text{Lie}(B))_{\text{gl}}^*$. By using the previous lemma, and the fact $\epsilon(x) = 1$, we have:

$$\begin{aligned} \xi_A^\sharp \left(x \triangleright \left(1_{\mathcal{U}(\text{Lie}(A))} + \sum_{n=1}^{+\infty} h^n a_n \right) \right) &= \xi_A^\sharp \left(\epsilon(x) 1_{\mathcal{U}(\text{Lie}(A))} + x \triangleright \sum_{n=1}^{+\infty} h^n a_n \right) = \left(1, \xi_A^b \left(x \triangleright \sum_{n=1}^{+\infty} h^n a_n \right) \right) \\ &= \left(1, \xi_B(x) \triangleright \xi_A^b \left(\sum_{n=1}^{+\infty} h^n a_n \right) \triangleleft \xi_B(x)^{-1} \right) \doteq \xi_B(x) \triangleright \left(1, \xi_A^b \left(\sum_{n=1}^{+\infty} h^n a_n \right) \right) \doteq \xi_B(x) \triangleright \xi_A^\sharp \left(1_{\mathcal{U}(\text{Lie}(A))} + \sum_{n=1}^{+\infty} h^n a_n \right). \end{aligned}$$

\square

2.5.6 The bare counit

Recall Majid's theorems 17 and 20. Let $\mathcal{H} = (\partial: I \rightarrow H, \rho)$ be a Hopf algebra crossed module. Consider $I^0 \doteq \ker(\epsilon: I \rightarrow \mathbb{C}) \subset I$. This is a non-unital bare algebra, closed under the action ρ of H on I , since I is an H -module coalgebra. This permits us to consider the bare algebra $\ker(\epsilon: I \rightarrow \mathbb{C}) \otimes_\rho H = I^0 \otimes_\rho H$. Recall that we have an inclusion $i_H: H \rightarrow I \otimes_\rho H$, with $i_H(x) = 1_I \otimes x$. If $v \otimes x \in I^0 \otimes H$ then, if $y \in H$, we have:

$$(v \otimes x)(1_H \otimes y) = \sum_{(x)} (vx' \triangleright 1_H) \otimes (x''y) = \sum_{(x)} (\epsilon(x')v) \otimes (x''y) = v \otimes (xy),$$

and

$$(1_H \otimes y)(v \otimes x) = \sum_{(y)} y' \triangleright v \otimes y''x.$$

Therefore, clearly:

Proposition 35. *Let $\mathcal{H} = (\partial: I \rightarrow H, \rho)$ be a Hopf algebra crossed module. We have a pre-crossed module:*

$$\widehat{\mathcal{BA}}(\mathcal{H}) = (t: I^0 \otimes_\rho H \rightarrow H, \triangleright, \triangleleft),$$

of bare algebras, where:

- $t(v \otimes x) = \partial(v)x$, where $v \in I^0$ and $x \in H$. Recall the target map $t: I \otimes_\rho H \rightarrow H$ of Lemma 20.
- $x \triangleright a = i_H(x)a$ and $a \triangleleft x = a i_H(x)$, where $a \in I^0 \otimes_\rho H$, $x \in H$, and $i_H: H \rightarrow I \otimes_\rho H$ is the inclusion.

By considering the reflection functor \mathcal{R} ; see 2.2.2, from the category of pre-crossed modules of bare algebras to the category of crossed modules of bare algebras, one therefore has a crossed module of bare algebras:

$$\mathcal{BA}(\mathcal{H}) = \mathcal{R}(\widehat{\mathcal{BA}}(\mathcal{H})) = (t: \underline{I^0 \otimes_\rho H} \rightarrow H, \triangleright, \triangleleft).$$

Suppose that $\mathcal{A} = (\partial: A \rightarrow B, \triangleright, \triangleleft)$ is a unital crossed module of bare algebras. Pass to the associated crossed module of Hopf algebras $\mathcal{U}(\text{Lie}(\mathcal{A})) = (\partial: \mathcal{U}(\text{Lie}(A)) \rightarrow \mathcal{U}(\text{Lie}(B)), \rho)$; 2.5.3. We therefore have a pre-crossed module of bare algebras $(t: \mathcal{U}^0(\text{Lie}(A)) \otimes_\rho \mathcal{U}(\text{Lie}(B)) \rightarrow \mathcal{U}(\text{Lie}(B)), \triangleright, \triangleleft)$.

Theorem 36 (bare counit). *Let $\kappa_{\mathcal{A}}: \mathcal{U}^0(\text{Lie}(A)) \otimes_\rho \mathcal{U}(\text{Lie}(B)) \rightarrow A$ be the linear map such that (see 2.5.3):*

$$\kappa_{\mathcal{A}}(v \otimes x) = \xi_A^b(v) \triangleleft \xi_B(x), \text{ for each } x \in \mathcal{U}(\text{Lie}(B)) \text{ and each } v \in \mathcal{U}(\text{Lie}(A)).$$

Then $\kappa_{\mathcal{A}}$ is an algebra map, and, in addition, the pair:

$$\hat{K}_{\mathcal{A}} = (\kappa_{\mathcal{A}}, \xi_B): \widehat{\mathcal{BA}}(\mathcal{U}(\text{Lie}(\mathcal{A}))) = (t: \mathcal{U}^0(\text{Lie}(A)) \otimes_\rho \mathcal{U}(\text{Lie}(B)) \rightarrow \mathcal{U}(\text{Lie}(B)), \triangleright, \triangleleft) \rightarrow (\partial: A \rightarrow B, \triangleright, \triangleleft)$$

is a map of pre-crossed modules of bare algebras $\hat{K}_{\mathcal{A}}: \widehat{\mathcal{BA}}(\mathcal{U}(\text{Lie}(\mathcal{A}))) \rightarrow \mathcal{A}$. Moreover, since \mathcal{A} is a unital crossed module of bare algebras, then $\hat{K}_{\mathcal{A}}$ descend to a crossed module map, the bare counit, denoted as:

$$K_{\mathcal{A}}: \mathcal{BA}(\mathcal{U}(\text{Lie}(\mathcal{A}))) \rightarrow \mathcal{A}.$$

Proof: Let $u \otimes x, v \otimes y \in \mathcal{U}^0(\text{Lie}(\mathcal{A})) \otimes_{\rho} \mathcal{U}(\text{Lie}(B))$. Suppose that $x \in B$, thus $\Delta(x) = x \otimes 1 + 1 \otimes x$. Then:

$$\begin{aligned} \kappa_{\mathcal{A}}((u \otimes x)(v \otimes y)) &= \sum_{(x)} \kappa_{\mathcal{A}}((u x' \triangleright v) \otimes (x'' y)) = \sum_{(x)} \xi_{\mathcal{A}}^b(u x' \triangleright v) \triangleleft \xi_B(x'' y) \\ &= \sum_{(x)} (\xi_{\mathcal{A}}^b(u) \xi_{\mathcal{A}}^b(x' \triangleright v)) \triangleleft (\xi_B(x'') \xi_B(y)) \\ &= \xi_{\mathcal{A}}^b(u) \xi_{\mathcal{A}}^b(x \triangleright v) \triangleleft \xi_B(y) + \xi_{\mathcal{A}}^b(u) \xi_B^b(v) \triangleleft \xi_B(x) \xi_B(y) \\ &= \xi_{\mathcal{A}}^b(u) (\xi_B(x) \triangleright \xi_{\mathcal{A}}^b(v) - \xi_{\mathcal{A}}^b(v) \triangleleft \xi_B(x)) \triangleleft \xi_B(y) + \xi_{\mathcal{A}}^b(u) \xi_{\mathcal{A}}^b(v) \triangleleft \xi_B(x) \xi_B(y) \\ &= (\xi_{\mathcal{A}}^b(u) \triangleleft \xi_B(x)) (\xi_{\mathcal{A}}^b(v) \triangleleft \xi_B(y)) = \kappa_{\mathcal{A}}(u \otimes x) \kappa_{\mathcal{A}}(v \otimes y), \end{aligned}$$

where we have used lemma 31 and the fact that the actions \triangleleft and \triangleright are fully interchangeable. Suppose, by induction, that we have proved that:

$$\kappa_{\mathcal{A}}((u \otimes x)(v \otimes y)) = (\xi_{\mathcal{A}}^b(u) \triangleleft \xi_B(x)) (\xi_{\mathcal{A}}^b(v) \triangleleft \xi_B(y)) = \kappa_{\mathcal{A}}(u \otimes x) \kappa_{\mathcal{A}}(v \otimes y),$$

in general, for all x of filtered degree $\leq n$ in $\mathcal{U}(\text{Lie}(B))$. If $b \in B$ we have, since $\Delta(b) = b \otimes 1 + 1 \otimes b$:

$$\begin{aligned} \kappa_{\mathcal{A}}((u \otimes xb)(v \otimes y)) &= \kappa_{\mathcal{A}}((u \otimes x)(1 \otimes b)(v \otimes y)) = \kappa_{\mathcal{A}}((u \otimes x)(b \triangleright v \otimes y) + (u \otimes x)(v \otimes by)) \\ &= \kappa_{\mathcal{A}}(u \otimes x) \kappa_{\mathcal{A}}(b \triangleright v \otimes y) + \kappa_{\mathcal{A}}(u \otimes x) \kappa_{\mathcal{A}}(v \otimes by) \\ &= (\xi_{\mathcal{A}}^b(u) \triangleleft \xi_B(x)) (\xi_{\mathcal{A}}^b(b \triangleright v) \triangleleft \xi_B(y)) + (\xi_{\mathcal{A}}^b(u) \triangleleft \xi_B(x)) (\xi_{\mathcal{A}}^b(v) \triangleleft \xi_B(by)) \\ &= (\xi_{\mathcal{A}}^b(u) \triangleleft \xi_B(x)) (\xi_B(b) \triangleright \xi_{\mathcal{A}}^b(v) - \xi_{\mathcal{A}}^b(v) \triangleleft \xi_B(b)) \triangleleft \xi_B(y) + (\xi_{\mathcal{A}}^b(u) \triangleleft \xi_B(x)) (\xi_{\mathcal{A}}^b(v) \triangleleft \xi_B(by)) \\ &= (\xi_{\mathcal{A}}^b(u) \triangleleft \xi_B(x)) \xi_B(b) \triangleright (\xi_{\mathcal{A}}^b(v) \triangleleft \xi_B(y)) = (\xi_{\mathcal{A}}^b(u) \triangleleft \xi_B(x)) \triangleleft \xi_B(b) (\xi_{\mathcal{A}}^b(v) \triangleleft \xi_B(y)) \\ &= (\xi_{\mathcal{A}}^b(u) \triangleleft \xi_B(xb)) (\xi_{\mathcal{A}}^b(v) \triangleleft \xi_B(y)) = \kappa_{\mathcal{A}}((u \otimes xb) \kappa_{\mathcal{A}}((v \otimes y))). \end{aligned}$$

Therefore $\kappa_{\mathcal{A}}$ is an algebra morphism. Now note that, if $y \in B$ and $v \otimes x \in \mathcal{U}^0(\text{Lie}(\mathcal{A})) \otimes_{\rho} \mathcal{U}(\text{Lie}(B))$:

$$\begin{aligned} \kappa_{\mathcal{A}}((v \otimes x) \triangleleft y) &\doteq \kappa_{\mathcal{A}}((v \otimes x)(1 \otimes y)) = \kappa_{\mathcal{A}}(v \otimes (xy)) = \xi_{\mathcal{A}}^b(v) \triangleleft \xi_B(xy) \\ &= (\xi_{\mathcal{A}}^b(v) \triangleleft \xi_B(x)) \triangleleft \xi(y) = \kappa_{\mathcal{A}}(v \otimes x) \triangleleft \xi_B(y). \end{aligned}$$

And, since $\Delta(y) = y \otimes 1 + 1 \otimes y$:

$$\begin{aligned} \kappa_{\mathcal{A}}(y \triangleright (v \otimes x)) &= \kappa_{\mathcal{A}}((1 \otimes y)(v \otimes x)) = \kappa_{\mathcal{A}}(y \triangleright v \otimes x + v \otimes yx) = \xi_{\mathcal{A}}^b(y \triangleright v) \triangleleft \xi_B(x) + \xi_{\mathcal{A}}^b(v) \otimes \xi_B(yx) \\ &= \xi(y) \triangleright \xi_{\mathcal{A}}^b(v) \triangleleft \xi_B(x) - \xi_{\mathcal{A}}^b(v) \triangleleft \xi_B(y) \xi_B(x) + \xi_{\mathcal{A}}^b(v) \otimes \xi_B(yx) \\ &= \xi_B(y) \triangleright \xi_{\mathcal{A}}^b(v) \triangleleft \xi_B(x) = \xi_B(y) \triangleright \kappa_{\mathcal{A}}(v \otimes x). \end{aligned}$$

Since $\mathcal{U}(\text{Lie}(B))$ is generated, as an algebra, by B , follows that $\hat{K}_{\mathcal{A}} = (\kappa_{\mathcal{A}}, \xi_{\mathcal{A}})$ preserves the actions \triangleleft and \triangleright . The rest of the assertions follow immediately. \square

We have an adjunction between the category of crossed modules of bare algebras and the category of crossed modules of Lie algebras; diagram (12), where Lie is the right adjoint. The bare counit K is its counit.

2.5.7 The crossed module inclusion

Let $\mathcal{H} = (\partial: I \rightarrow H, \rho)$ be a crossed module of Hopf algebras. Consider the crossed modules of groups $\mathcal{H}_{\text{gl}}^*$ and $\mathcal{BA}(\mathcal{H})_{\bullet}^*$; see 2.2.4, 2.4.4 and 2.5.6. We have a crossed module map, called the crossed module inclusion:

$$\text{Inc}_{\mathcal{H}} = (J_{\mathcal{H}}, \iota_H): (\mathcal{H})_{\text{gl}}^* \rightarrow (\mathcal{BA}(\mathcal{H}))_{\bullet}^*.$$

The map $\iota_{\mathcal{H}}: H_{\text{gl}}^* \rightarrow H_{\bullet}^*$ is simply group inclusion. Noting that $\widehat{\mathcal{BA}}(\mathcal{H}) = (t: I^0 \otimes_{\rho} H \rightarrow H, \triangleright, \triangleleft)$, where $I^0 \otimes_{\rho} H$ is a subalgebra of the Hopf algebra $I \otimes_{\rho} H$, being that I is a Hopf subalgebra of $I \otimes_{\rho} H$, follows that we have group inclusions:

$$I_{\text{gl}}^* \rightarrow (I \otimes_{\rho} H)_{\text{gl}}^* \rightarrow (I \otimes_{\rho} H)^* = (I^0 \otimes_{\rho} H)_{\bullet}^*.$$

The group map $J_{\mathcal{H}}: I_{\text{gl}}^* \rightarrow \underline{(I^0 \otimes_{\rho} H)}_{\bullet}^*$, is obtained by composing with the map $(I^0 \otimes_{\rho} H)_{\bullet}^* \rightarrow \underline{(I^0 \otimes_{\rho} H)}_{\bullet}^*$, induced by the algebra projection $I^0 \otimes_{\rho} H \rightarrow \underline{I^0 \otimes_{\rho} H}$. Clearly $\text{Inc}_{\mathcal{H}}: (\mathcal{H})_{\text{gl}}^* \rightarrow (\mathcal{BA}(\mathcal{H}))_{\bullet}^*$ is a map of crossed modules of groups.

The crossed module counit $\text{Proj}_{\mathcal{A}}: \mathcal{U}(\text{Lie}(\mathcal{A}))_{\text{gl}}^* \rightarrow \mathcal{A}_{\bullet}^*$ (Theorem 34) can be obtained from the bare counit $K_{\mathcal{A}}: \mathcal{BA}(\mathcal{U}(\text{Lie}(\mathcal{A}))) \rightarrow \mathcal{A}$ and the crossed module inclusion as the composition; c.f. (12), (13) and (14):

$$\mathcal{U}(\text{Lie}(\mathcal{A}))_{\text{gl}}^* \xrightarrow{\text{Inc}_{\mathcal{U}(\text{Lie}(\mathcal{A}))}} (\mathcal{BA}(\mathcal{U}(\text{Lie}(\mathcal{A}))))_{\bullet}^* \xrightarrow{(K_{\mathcal{A}})_{\bullet}^*} \mathcal{A}_{\bullet}^*.$$

3 Holonomy over Hopf and bare algebras

3.1 Some facts on iterated integrals

3.1.1 Differentiating and integrating over general algebras

We will consider algebras H without any topology in it, justifying the need for the discussion now. Recall that all normed spaces of finite dimension are equivalent to some \mathbb{R}^n , and any linear map $\mathbb{R}^n \rightarrow \mathbb{R}^m$ is C^{∞} .

Definition 37 (Smooth map, partial differentiation and integration). *Let H be an algebra over \mathbb{C} . Given an open or closed set (usually a simplex or a parallelepiped) $U \subset \mathbb{R}^n$, a map $f: U \rightarrow H$ is called smooth if:*

1. *The range of f is contained in a finite dimensional subspace V of H .*
2. *The induced map $f: U \rightarrow V$ is smooth (in other words C^{∞}).*

Let $C^{\infty}(U, H)$ be the vector space of smooth maps $f: U \rightarrow H$. Given that the range of each $f \in C^{\infty}(U, H)$ is contained in a finite dimensional vector space $V \subset H$, we can consider the partial differentiation operators $\frac{\partial}{\partial x_i}: C^{\infty}(U, H) \rightarrow C^{\infty}(U, H)$, as well as, whenever U is compact, the integration $\int_U \cdots \int f(x) d^n x \in V \subset H$.

Note that products and sums of smooth functions $U \rightarrow H$ are smooth.

Given that linear maps between finite dimensional vector spaces are always smooth it follows that:

Lemma 38. *If $f: U \rightarrow H$ is smooth and $T: H \rightarrow H'$ is a linear map then $T \circ f: U \rightarrow H'$ is smooth. Moreover T commutes with partial differentiation and integration.*

Lemma 39 (Uniqueness of integration). *Let H be an algebra over \mathbb{C} . Let $f: [a, b] \rightarrow H$ be a smooth map. Given $g \in H$ there exists a unique smooth function $F: [a, b] \rightarrow H$ such that $F'(x) = f(x), \forall x \in [a, b]$, and with $f(a) = g$. In fact $F(t) = g + \int_a^t f(u) du, \forall t \in [a, b]$.*

Proof: Let $W \subset H$ be a finite dimensional vector space containing both $f([a, b])$ and $\{g\}$. Then use uniqueness of integration for smooth functions with values in finite dimensional vector spaces. \square

3.1.2 Definition of iterated integrals

We present a small part of the theory of iterated integrals appearing in [Che73]. Let H be an algebra. Given smooth functions $u \in [0, 1] \mapsto f_i(u) \in H$, where $i = 1, 2, \dots, n$, we define, given $t_0, t \in [0, 1]$, with $t_0 \leq t$:

$$\oint_{t_0}^t f_1(u) du = \int_{t_0}^t f_1(u_1) du_1.$$

In general, we put:

$$\begin{aligned} \oint_{t_0}^t f_1(u) * \cdots * f_n(u) du &= \int_{t_0}^t \int_{t_0}^{u_1} \int_{t_0}^{u_2} \int_{t_0}^{u_3} \cdots \int_{t_0}^{u_{n-1}} f_1(u_1) f_2(u_2) f_3(u_3) \cdots f_n(u_n) du_n \cdots du_3 du_2 du_1 \\ &= \int_{\{t \geq u_1 \geq u_2 \geq \cdots \geq u_n \geq t_0\}} \cdots \int f_1(u_1) f_2(u_2) f_3(u_3) \cdots f_n(u_n) du_n \cdots du_3 du_2 du_1. \end{aligned}$$

Clearly:

$$\frac{d}{dt} \oint_{t_0}^t f_1(u) * f_2(u) \cdots * f_n(u) du = f_1(t) \oint_{t_0}^t f_2(u) \cdots * f_n(u) du. \quad (34)$$

We will also consider the following alternative presentation of iterated integrals, which, for reasons which will be clearer later, we prefer to denote differently. Given $t_1, t \in [0, 1]$, with $t \leq t_1$, we put:

$$\oint_t^{t_1} f_1(u) du = \int_t^{t_1} f_1(u_1) du_1,$$

and, in general:

$$\begin{aligned} \oint_t^{t_1} f_1(u) * \cdots * f_n(u) du &= \int_t^{t_1} \int_{u_1}^{t_1} \int_{u_2}^{t_1} \int_{u_3}^{t_1} \cdots \int_{u_{n-1}}^{t_1} f_1(u_n) f_2(u_{n-1}) f_3(u_{n-2}) \cdots f_n(u_1) du_n \cdots du_3 du_2 du_1 \\ &= \int_{\{t \leq u_1 \leq u_2 \leq \cdots \leq u_n \leq t_1\}} f_1(u_n) f_2(u_{n-1}) f_3(u_{n-2}) \cdots f_n(u_1) du_n \cdots du_3 du_2 du_1. \end{aligned}$$

We have:

$$\frac{d}{dt} \oint_t^{t_1} f_1(u) * \cdots * f_n(u) du = - \left(\oint_t^{t_1} f_1(u) * \cdots * f_{n-1}(u) du \right) f_n(t).$$

Clearly (by inverting the order of integration, and then renaming the variables.)

Lemma 40. *For all $0 < a < b < 1$ we have:*

$$\oint_a^b f_1(u) * \cdots * f_n(u) du = \oint_a^b f_1(u) * \cdots * f_n(u) du.$$

3.2 The holonomy of algebra valued connections

3.2.1 1-paths, 2-paths and 3-paths. Thin 2-paths.

Nothing here is new; see for example [CP94, SW11, SW13, FP10, FP11b]. Let M be a smooth manifold. We denote by $\mathcal{P}^1(M)$ the space of continuous and piecewise smooth maps $\gamma: [0, 1] \rightarrow M$, called 1-paths, and normally represented in the form below, emphasizing initial and end points:

$$x = \gamma(0) \xrightarrow{\gamma} \gamma(1) = y.$$

If γ_1 and γ_2 are 1-paths with $\gamma_1(1) = \gamma_2(0)$, we define their concatenation, written as:

$$\left(x \xrightarrow{\gamma_1} y \xrightarrow{\gamma_2} z \right) = x \xrightarrow{\gamma_1 \gamma_2} z,$$

where $x = \gamma_1(0)$, $y = \gamma_1(1) = \gamma_2(0)$ and $z = \gamma_2(1)$, in the usual way:

$$(\gamma_1 \gamma_2)(t) = \begin{cases} \gamma_1(2t), & t \in [0, 1/2] \\ \gamma_2(2t - 1), & t \in [1/2, 1] \end{cases}$$

The reversion of the 1-path $x \xrightarrow{\gamma} y$ is the 1-path $y \xrightarrow{\gamma^{-1}} x$, where $\gamma^{-1}(t) = \gamma(1 - t), \forall t \in [0, 1]$. The composition of 1-paths is not associative. This will not pose any problem for this paper. This can be fixed, for example, by considering a thin homotopy relation between 1-paths; see [CP94] and below.

We analogously define a 2-path as being a map $\Gamma: [0, 1]^2 \rightarrow M$, which is to be continuous and piecewise smooth for some paving of the square by polygons, transverse to the boundary of the square. Moreover, we suppose that $\Gamma(\{0, 1\} \times [0, 1])$ has at most two elements. Putting $\{x\} = \Gamma(\{0\} \times [0, 1])$, $\{y\} = \Gamma(\{1\} \times [0, 1])$ (the left and right boundaries of Γ), and also $\gamma_1(t) = \Gamma(t, 1)$, $\gamma_0(t) = \Gamma(t, 0)$, where $t \in [0, 1]$ (the bottom

and top boundaries of Γ), defining 1-paths $\gamma_0, \gamma_1: [0, 1] \rightarrow M$, we denote Γ in the form $\Gamma = x \begin{array}{c} \xrightarrow{\gamma_1} \\ \uparrow \Gamma \\ \xrightarrow{\gamma_0} \end{array} y$,

emphasizing its boundaries. The set of 2-paths in M is denoted by $\mathcal{P}^2(M)$.

Clearly 2-paths compose horizontally and vertically in the obvious way, whenever pertinent boundaries coincide. We also have whiskerings of 2-paths by 1-paths: consider a 2-path Γ , and 1-paths γ_1 and γ_2 , such that the left boundary of Γ is $\gamma_1(1)$ and the right boundary of Γ is $\gamma_2(0)$; see below. We put:

$$\begin{aligned}
& x \begin{array}{c} \xrightarrow{\phi_0} \\ \uparrow \Gamma \\ \xrightarrow{\phi_1} \end{array} y \xrightarrow{\gamma_2} z = x \begin{array}{c} \xrightarrow{\phi_1 \gamma_2} \\ \uparrow \Gamma \gamma_2 \\ \xrightarrow{\phi_1 \gamma_2} \end{array} z, \text{ where } (\Gamma \gamma_2)(t, s) = \begin{cases} \Gamma(2t, s), & \text{if } 0 \leq t \leq 1/2, \\ \gamma_2(2t - 1), & \text{if } 1/2 \leq t \leq 1 \end{cases}, \\
& w \xrightarrow{\gamma_1} x \begin{array}{c} \xrightarrow{\phi_1} \\ \uparrow \Gamma \\ \xrightarrow{\phi_0} \end{array} y = w \begin{array}{c} \xrightarrow{\gamma_1 \phi_1} \\ \uparrow \gamma_1 \Gamma \\ \xrightarrow{\gamma_1 \phi_0} \end{array} y, \text{ where } (\gamma_1 \Gamma)(t, s) = \begin{cases} \gamma_1(2t), & \text{if } 0 \leq t \leq 1/2, \\ \Gamma(2t - 1, s), & \text{if } 1/2 \leq t \leq 1 \end{cases}.
\end{aligned}$$

These compositions are not associative and do not satisfy the interchange law. This can be fixed, in order to define a 2-category, by considering thin homotopy relations between 2-paths; see [FP10, FP11a, SW11].

Definition 41 (Thin 2-path. Thin homotopy). A 2-path $x \begin{array}{c} \xrightarrow{\gamma_2} \\ \uparrow \Gamma \\ \xrightarrow{\gamma_2} \end{array} y$ is called thin if, at each $(s, t) \in [0, 1]^2$, where $\Gamma: [0, 1]^2 \rightarrow M$ is smooth, the rank of the derivative of Γ is ≤ 1 . Such a 2-path Γ is called a thin homotopy from γ_0 to γ_1 , and the 1-paths γ_0 and γ_1 are said to be thin-homotopic.

Remark 42 (The thin homotopy groupoid of a smooth manifold). Thin homotopy is an equivalence relation, the composition of 1-paths descends to the quotient, and we have a groupoid, the thin homotopy groupoid of M , whose set of objects is M and whose morphisms are 1-paths up to thin-homotopy; see [FP10, CP94].

Definition 43 (3-path. Thin 3-path.). Consider two 2-paths $x \begin{array}{c} \xrightarrow{\gamma_1} \\ \uparrow \Gamma \\ \xrightarrow{\gamma_0} \end{array} y$ and $x \begin{array}{c} \xrightarrow{\phi_1} \\ \uparrow \Phi \\ \xrightarrow{\phi_0} \end{array} y$. A 3-path from the first to the second is given by a map $J: [0, 1]^3 \rightarrow M$, such that:

- The map J is piecewise smooth for some paving of the cube $[0, 1]^3$ by polygons, which is transverse to the boundary of the cube.
- $J(\{0\} \times [0, 1]^2) = \{x\}$ and $J(\{1\} \times [0, 1]^2) = \{y\}$.
- $J(t, s, 0) = \Gamma(t, s)$ and $J(t, s, 1) = \Phi(t, s)$, for all $t, s \in [0, 1]$.

Each 3-path J has top and bottom 2-paths J^t and J^b , where $J^t(t, x) = J(t, 1, x)$ and $J^b(t, x) = J(t, 0, x)$, where $t, x \in [0, 1]$. We say that J is thin if both J^t and J^b are thin 2-paths, and also, given any $(t, s, x) \in [0, 1]^3$ where $J: [0, 1]^3 \rightarrow M$ is smooth, the derivative of J at (t, s, x) has rank ≤ 2 .

Remark 44 (The thin homotopy 2-groupoid of a smooth manifold). Thin homotopy of 2-paths is an equivalence relation. All operations already defined for 1-paths and 2-paths (whiskerings and horizontal/vertical compositions) descend to the quotients by thin homotopy. Moreover, we have a 2-groupoid, the thin homotopy 2-groupoid of M , with objects being the points of M , with morphisms being the 1-paths connecting points, up to thin homotopy, and 2-morphisms being 2-paths, up to thin homotopy. We note that this result is not immediate and the details appear spelled out (using two different arguments) in [FP10] and [FP11a].

3.2.2 The holonomy of a 1-connection

Definition 45 (Algebra-valued smooth form on a manifold). Let M be a smooth manifold. Let H be a bare algebra. A smooth n -form ω is an n -form in M , with values in H , whose range is included in a finite dimensional subspace $V \subset H$, being that ω is smooth when considered to have values in V .

We consider holonomy operators with values in $H[[h]]$, where H is an algebra. This is due to Chen [Che73].

Definition 46. Let U be an open or closed set of \mathbb{R}^n . A function $x \in U \mapsto \sum_{n=1}^{+\infty} f_n(x)h^n \in H[[h]]$ is said to be smooth if each function $f_n: U \rightarrow H$ is smooth. Differentiation and integration of smooth functions will be done term-wise, in the obvious way.

Lemma 47. Let H be an algebra. Let $x \in H[[h]]$. Let $f: [0, 1] \rightarrow H[[h]]$ be a smooth function. The initial value problems below have unique smooth solutions, $g_1, g_2: [0, 1] \rightarrow H[[h]]$:

$$\begin{aligned} g_1'(t) &= h g_1(t) f(t); & \text{with } g_1(0) &= x, \\ g_2'(t) &= h f(t) g_2(t); & \text{with } g_2(0) &= x. \end{aligned}$$

Proof: Follows directly by induction and by uniqueness of integration; Lemma 39. Recall that products and sums of smooth functions $[0, 1] \rightarrow H$ are smooth. \square

Definition 48 ((local) connection and curvature. Primitive connection). Let H be a unital algebra, with unit 1_H . A (local) connection, or local connection form, in M is a smooth 1-form ω in M with values in H . If H is a Hopf algebra, a connection ω is called primitive if it takes values inside the primitive space $\text{Prim}(H)$ of H . Given a connection ω its curvature (note our conventions) is the H -valued smooth 2-form:

$$\mathcal{F}_\omega = d\omega - \frac{1}{2}\omega \wedge \omega.$$

Here, if X and Y are vector fields in M , we put

$$\frac{1}{2}(\omega \wedge^{[1]} \omega)(X, Y) = \frac{1}{2}(\omega \wedge \omega)(X, Y) = [\omega(X), \omega(Y)].$$

From a connection form ω we can build the graded connection form $\bar{\omega} \doteq h\omega$, and the graded curvature 2-form:

$$\mathcal{F}_{\bar{\omega}} \doteq h d\omega - h^2 \frac{1}{2}\omega \wedge \omega = d\bar{\omega} - \frac{1}{2}\bar{\omega} \wedge \bar{\omega}.$$

Let $\gamma: [0, 1] \rightarrow M$ be a piecewise smooth curve in M , a smooth manifold. Given an H -valued connection form in M , where H is a unital algebra, consider the following $H[[h]]$ valued function in $[0, 1]$, where γ' denotes the derivative of γ . Here $t, t_0 \in [0, 1]$:

$$P_{\bar{\omega}}(\gamma, [t, t_0]) = \sum_{n=0}^{+\infty} h^n \oint_{t_0}^t \underbrace{\omega(\gamma'(u)) * \cdots * \omega(\gamma'(u))}_{n \text{ terms}} du = \sum_{n=0}^{+\infty} \oint_{t_0}^t \underbrace{\bar{\omega}(\gamma'(u)) * \cdots * \bar{\omega}(\gamma'(u))}_{n \text{ terms}} du.$$

Here we have conventioned that $\oint_{t_0}^t \underbrace{\omega(\gamma'(u)) * \cdots * \omega(\gamma'(u))}_{0 \text{ terms}} du = 1_H$, where $t, t_0 \in [0, 1]$. From Lemma 39:

Proposition 49. The function $t \mapsto P_{\bar{\omega}}(\gamma, [t, t_0])$ is the unique smooth solution of the differential equation:

$$f'(t) = \bar{\omega}(\gamma'(t)) f(t), \quad f(t_0) = 1_H.$$

In general we have, for $t, t_0 \in [0, 1]$ (note Lemma 40):

$$\frac{\partial}{\partial t} P_{\bar{\omega}}(\gamma, [t, t_0]) = \bar{\omega}(\gamma'(t)) P_{\bar{\omega}}(\gamma, [t, t_0]), \quad \frac{\partial}{\partial t_0} P_{\bar{\omega}}(\gamma, [t, t_0]) = -P_{\bar{\omega}}(\gamma, [t, t_0]) \bar{\omega}(\gamma'(t_0)). \quad (35)$$

Standard iterated integral theory (or simple calculations) yields.

Proposition 50. Let γ_1 and γ_2 be piecewise smooth curves in M , such that $\gamma_1(1) = \gamma_2(0)$. Let $\gamma = \gamma_1 \gamma_2$ be the concatenation of γ_1 and γ_2 . Then, for each $t \in [1/2, 1]$, we have:

$$P_{\bar{\omega}}(\gamma, [t, 0]) = P_{\bar{\omega}}(\gamma_2, [2t - 1, 0]) P_{\bar{\omega}}(\gamma_1, [1, 0]).$$

Proposition 51. Let γ be a piecewise smooth curve in M . Then, if $0 \leq t_0 \leq s \leq t_1 \leq 1$, we have:

$$P_{\bar{\omega}}(\gamma, [t_1, t_0]) = P_{\bar{\omega}}(\gamma, [t_1, s]) P_{\bar{\omega}}(\gamma, [s, t_0]). \quad (36)$$

Furthermore, this is valid for arbitrary $t_0, t_1, s \in [0, 1]$.

The following is well known.

Lemma 52. *Suppose that H is an Hopf algebra and that ω is a primitive connection. Then $t \mapsto P_{\overline{\omega}}(\gamma, [t, t_0])$ takes values in H_{gl}^* (the space of grouplike elements of $H[[h]]$ whose power series starts at 1_H ; see 2.4.4.)*

Proof: The map $t \mapsto \Delta(P_{\overline{\omega}}(\gamma, [t, t_0])) \in (H \otimes H)[[h]]$ is smooth. Moreover $\Delta(P_{\overline{\omega}}(\gamma, [t_0, t_0])) = \Delta(1_H) = 1_H \otimes 1_H$. Also:

$$\begin{aligned} \frac{d}{dt} \Delta(P_{\overline{\omega}}(\gamma, [t, t_0])) &= \Delta \left(\frac{d}{dt} P_{\overline{\omega}}(\gamma, [t, t_0]) \right) = \Delta(\overline{\omega}(\gamma'(t)) P_{\overline{\omega}}(\gamma, [t, t_0])) \\ &= \Delta(\overline{\omega}(\gamma'(t))) \Delta(P_{\overline{\omega}}(\gamma, [t, t_0])) = (\overline{\omega}(\gamma'(t)) \otimes 1_H + 1_H \otimes \overline{\omega}(\gamma'(t))) \Delta(P_{\overline{\omega}}(\gamma, [t, t_0])). \end{aligned}$$

The map $t \mapsto P_{\overline{\omega}}(\gamma, [t, t_0]) \otimes P_{\overline{\omega}}(\gamma, [t, t_0])$ is smooth, and $P_{\overline{\omega}}(\gamma, [t_0, t_0]) \otimes P_{\overline{\omega}}(\gamma, [t_0, t_0]) = 1_H \otimes 1_H$. Moreover:

$$\begin{aligned} \frac{d}{dt} (P_{\overline{\omega}}(\gamma, [t, t_0]) \otimes P_{\overline{\omega}}(\gamma, [t, t_0])) &= \left(\frac{d}{dt} P_{\overline{\omega}}(\gamma, [t, t_0]) \right) \otimes P_{\overline{\omega}}(\gamma, [t, t_0]) + P_{\overline{\omega}}(\gamma, [t, t_0]) \otimes \left(\frac{d}{dt} P_{\overline{\omega}}(\gamma, [t, t_0]) \right) \\ &= \left(\omega(\gamma'(t)) P_{\overline{\omega}}(\gamma, [t, t_0]) \right) \otimes P_{\overline{\omega}}(\gamma, [t, t_0]) + P_{\overline{\omega}}(\gamma, [t, t_0]) \otimes \left(\omega(\gamma'(t)) P_{\overline{\omega}}(\gamma, [t, t_0]) \right) \\ &= (\omega(\gamma'(t)) \otimes 1_H + 1_H \otimes \omega(\gamma'(t))) (P_{\overline{\omega}}(\gamma, [t, t_0]) \otimes P_{\overline{\omega}}(\gamma, [t, t_0])). \end{aligned}$$

Now apply Lemma 47 to conclude that $\Delta(P_{\overline{\omega}}(\gamma, [t, t_0])) = P_{\overline{\omega}}(\gamma, [t, t_0]) \otimes P_{\overline{\omega}}(\gamma, [t, t_0]), \forall t \in [t_0, 1]$. \square

The following frequently used compatibility between holonomy and algebra maps has a trivial proof.

Lemma 53. *Let $f: H \rightarrow H'$ be an algebra map. Let $\omega \in \Omega^1(M, H)$. We have, for each $t_0, t_1 \in [0, 1]$.*

$$f(P_{\overline{\omega}}(\gamma, [t_1, t_0])) = P_{f(\overline{\omega})}(\gamma, [t_1, t_0]).$$

3.2.3 The curvature and holonomy

Given a manifold M , let now $\Gamma: [0, 1]^2 \rightarrow M$ be a 2-path. Therefore $s \mapsto \Gamma(0, s)$ and $s \mapsto \Gamma(1, s)$ are constant 1-paths. Define, given $(t, s) \in [0, 1] \times [0, 1]$:

$$P_{\overline{\omega}}(\Gamma, [t, 0], s) = \sum_{n=0}^{+\infty} h^n \oint_0^t \underbrace{\omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right) * \cdots * \omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right)}_{n \text{ terms}} du.$$

If $s \in [0, 1]$, considering the 1-path γ_s with $\gamma_s(t) = \Gamma(s, t), \forall s, t \in [0, 1]$, we have $P_{\overline{\omega}}(\Gamma, [t, 0], s) = P_{\overline{\omega}}(\gamma_s, [t, 0])$. For each $s, t \in [0, 1]$ we have:

$$\begin{aligned} \frac{\partial}{\partial s} P_{\overline{\omega}}(\Gamma, [t, 0], s) &= \sum_{n=0}^{+\infty} h^n \\ &\sum_{i+j+1=n} \oint_0^t \underbrace{\omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right) * \cdots * \omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right)}_{i \text{ terms}} * \frac{\partial}{\partial s} \omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right) * \underbrace{\omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right) * \cdots * \omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right)}_{j \text{ terms}} du. \end{aligned}$$

Note that:

$$\frac{\partial}{\partial s} \omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right) = d\omega \left(\frac{\partial}{\partial s} \Gamma(u, s), \frac{\partial}{\partial u} \Gamma(u, s) \right) + \frac{\partial}{\partial u} \omega \left(\frac{\partial}{\partial s} \Gamma(u, s) \right).$$

Therefore (integrating by parts, and noting that $\frac{\partial}{\partial s} \Gamma(0, s) = 0$):

$$\begin{aligned} \frac{\partial}{\partial s} P_{\overline{\omega}}(\Gamma, [t, 0], s) &= \sum_{n=0}^{+\infty} h^n \sum_{i+j+1=n} \left(\right. \\ &\oint_0^t \underbrace{\omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right) * \cdots * \omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right)}_{i \text{ terms}} * d\omega \left(\frac{\partial}{\partial s} \Gamma(u, s), \frac{\partial}{\partial u} \Gamma(u, s) \right) * \underbrace{\omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right) * \cdots * \omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right)}_{j \text{ terms}} du \\ &\left. - \oint_0^t \underbrace{\omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right) * \cdots * \omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right)}_{i \text{ terms}} * \underbrace{\left(\omega \left(\frac{\partial}{\partial s} \Gamma(u, s) \right) \omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right) \right) * \omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right) * \cdots * \omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right)}_{j \text{ terms}} du \right) \end{aligned}$$

$$\begin{aligned}
& + \oint_0^t \underbrace{\omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right) * \cdots * \omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right) * \left(\omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right) \omega \left(\frac{\partial}{\partial s} \Gamma(u, s) \right) \right)}_{i \text{ terms}} * \underbrace{\omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right) * \cdots * \omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right)}_{j \text{ terms}} du \\
& + \sum_{n=0}^{+\infty} h^n \omega \left(\frac{\partial}{\partial s} \Gamma(t, s) \right) \oint_0^t \underbrace{\omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right) * \cdots * \omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right)}_{(n-1) \text{ terms}} du.
\end{aligned}$$

Thus:

$$\begin{aligned}
\frac{\partial}{\partial s} P_{\overline{\omega}}(\Gamma, [t, 0], s) & = \sum_{n=0}^{+\infty} h^n \sum_{i+j+1=n} \\
& \oint_0^t \underbrace{\omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right) * \cdots * \omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right)}_{i \text{ terms}} * d\omega \left(\frac{\partial}{\partial s} \Gamma(u, s), \frac{\partial}{\partial u} \Gamma(u, s) \right) * \underbrace{\omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right) * \cdots * \omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right)}_{j \text{ terms}} du \\
& - \sum_{n=0}^{+\infty} h^n \sum_{p+q+2=n} \\
& \oint_0^t \underbrace{\omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right) * \cdots * \omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right)}_{p \text{ terms}} * \frac{1}{2} \omega \wedge \omega \left(\frac{\partial}{\partial s} \Gamma(u, s), \frac{\partial}{\partial u} \Gamma(u, s) \right) * \underbrace{\omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right) * \cdots * \omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right)}_{q \text{ terms}} du \\
& + \sum_{n=0}^{+\infty} h^n \omega \left(\frac{\partial}{\partial s} \Gamma(t, s) \right) \oint_0^t \underbrace{\omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right) * \cdots * \omega \left(\frac{\partial}{\partial u} \Gamma(u, s) \right)}_{(n-1) \text{ terms}} du \\
& = \sum_{i,j=0}^{+\infty} \left(\oint_0^t \underbrace{\overline{\omega} \left(\frac{\partial}{\partial u} \Gamma(u, s) \right) * \cdots * \overline{\omega} \left(\frac{\partial}{\partial u} \Gamma(u, s) \right)}_{i \text{ terms}} * \mathcal{F}_{\overline{\omega}} \left(\frac{\partial}{\partial s} \Gamma(u, s), \frac{\partial}{\partial u} \Gamma(u, s) \right) * \underbrace{\overline{\omega} \left(\frac{\partial}{\partial u} \Gamma(u, s) \right) * \cdots * \overline{\omega} \left(\frac{\partial}{\partial u} \Gamma(u, s) \right)}_{j \text{ terms}} du \right) \\
& + \sum_{n=0}^{+\infty} \overline{\omega} \left(\frac{\partial}{\partial s} \Gamma(t, s) \right) \oint_0^t \underbrace{\overline{\omega} \left(\frac{\partial}{\partial u} \Gamma(u, s) \right) * \cdots * \overline{\omega} \left(\frac{\partial}{\partial u} \Gamma(u, s) \right)}_{(n-1) \text{ terms}} du = A + B.
\end{aligned}$$

Now note that:

$$\begin{aligned}
A & = \sum_{i \in \mathbb{N}} \oint_0^t \underbrace{\overline{\omega} \left(\frac{\partial}{\partial u} \Gamma(u, s) \right) * \cdots * \overline{\omega} \left(\frac{\partial}{\partial u} \Gamma(u, s) \right)}_{i \text{ terms}} * \left(\mathcal{F}_{\overline{\omega}} \left(\frac{\partial}{\partial s} \Gamma(u, s), \frac{\partial}{\partial u} \Gamma(u, s) \right) P_{\overline{\omega}}(\Gamma, u, s) \right) du \\
& = \sum_{i \in \mathbb{N}} \oint_0^t \underbrace{\overline{\omega} \left(\frac{\partial}{\partial u} \Gamma(u, s) \right) * \cdots * \overline{\omega} \left(\frac{\partial}{\partial u} \Gamma(u, s) \right)}_{i \text{ terms}} * \left(\mathcal{F}_{\overline{\omega}} \left(\frac{\partial}{\partial s} \Gamma(u, s), \frac{\partial}{\partial u} \Gamma(u, s) \right) P_{\overline{\omega}}(\Gamma, u, s) \right) du \\
& = \int_0^t P_{\overline{\omega}}(\Gamma, [t, u], s) \mathcal{F}_{\overline{\omega}} \left(\frac{\partial}{\partial s} \Gamma(u, s), \frac{\partial}{\partial u} \Gamma(u, s) \right) P_{\overline{\omega}}(\Gamma, [u, 0], s) du.
\end{aligned}$$

The first and third equations are tautological. The second follows from Lemma 40. This well know result follows:

Lemma 54. *In the conditions stated in the beginning of 3.2.3, we have, for each $t, s \in [0, 1]$:*

$$\begin{aligned}
& \frac{\partial}{\partial s} P_{\overline{\omega}}(\Gamma, [t, 0], s) \\
& = \int_0^t P_{\overline{\omega}}(\Gamma, [t, u], s) \mathcal{F}_{\overline{\omega}} \left(\frac{\partial}{\partial s} \Gamma(u, s), \frac{\partial}{\partial u} \Gamma(u, s) \right) P_{\overline{\omega}}(\Gamma, [u, 0], s) du + \overline{\omega} \left(\frac{\partial}{\partial s} \Gamma(t, s) \right) P_{\overline{\omega}}(\Gamma, [t, 0], s). \quad (37)
\end{aligned}$$

In particular, since $\frac{\partial}{\partial s} \Gamma(1, s) = 0, \forall s \in [0, 1]$, because Γ is a 2-path:

$$\frac{d}{ds} P_{\overline{\omega}}(\Gamma, [1, 0], s) = \int_0^1 P_{\overline{\omega}}(\Gamma, [1, u], s) \mathcal{F}_{\overline{\omega}} \left(\frac{\partial}{\partial s} \Gamma(u, s), \frac{\partial}{\partial u} \Gamma(u, s) \right) P_{\overline{\omega}}(\Gamma, [u, 0], s) du, \quad (38)$$

or, by (36),

$$\frac{d}{ds} P_{\overline{\omega}}(\Gamma, [1, 0], s) = \left(\int_0^1 P_{\overline{\omega}}(\Gamma, [1, u], s) \mathcal{F}_{\overline{\omega}} \left(\frac{\partial}{\partial s} \Gamma(u, s), \frac{\partial}{\partial u} \Gamma(u, s) \right) P_{\overline{\omega}}(\Gamma, [1, u], s)^{-1} du \right) P_{\overline{\omega}}(\Gamma, [1, 0], s). \quad (39)$$

In particular it follows that thin homotopic 2-paths have the same holonomy, a fact observed in [CP94].

Since $\frac{d}{ds} (P_{\overline{\omega}}(\Gamma, [1, 0], s) P_{\overline{\omega}}(\Gamma, [1, 0], s)^{-1}) = 0$, using Leibniz law, $\frac{\partial}{\partial s} (a(s) b(s)) = (\frac{\partial}{\partial s} a(s)) b(s) + a(s) (\frac{\partial}{\partial s} b(s))$:

$$\frac{d}{ds} P_{\overline{\omega}}(\Gamma, [0, 1], s) = - \left(\int_0^1 P_{\overline{\omega}}(\Gamma, [0, u], s) \mathcal{F}_{\overline{\omega}} \left(\frac{\partial}{\partial s} \Gamma(u, s), \frac{\partial}{\partial u} \Gamma(u, s) \right) P_{\overline{\omega}}(\Gamma, [0, u], s)^{-1} du \right) P_{\overline{\omega}}(\Gamma, [0, 1], s). \quad (40)$$

(Note $P_{\overline{\omega}}(\Gamma, [1, 0], s)^{-1} = P_{\overline{\omega}}(\Gamma, [1, 0], s)$.) Also, proceeding in the same way as before:

$$\begin{aligned} \frac{\partial}{\partial s} P_{\overline{\omega}}(\Gamma, [0, t], s) = & - \left(\int_0^t P_{\overline{\omega}}(\Gamma, [0, u], s) \mathcal{F}_{\overline{\omega}} \left(\frac{\partial}{\partial s} \Gamma(u, s), \frac{\partial}{\partial u} \Gamma(u, s) \right) P_{\overline{\omega}}(\Gamma, [u, 0], s) \right) P_{\overline{\omega}}(\Gamma, [0, t], s) \\ & - P_{\overline{\omega}}(\Gamma, [0, t], s) \overline{\omega} \left(\frac{\partial}{\partial s} \Gamma(t, s) \right). \end{aligned} \quad (41)$$

$$\begin{aligned} \frac{\partial}{\partial s} P_{\overline{\omega}}(\Gamma, [t, 1], s) = & - \int_t^1 P_{\overline{\omega}}(\Gamma, [t, u], s) \mathcal{F}_{\overline{\omega}} \left(\frac{\partial}{\partial s} \Gamma(u, s), \frac{\partial}{\partial u} \Gamma(u, s) \right) P_{\overline{\omega}}(\Gamma, [u, 1], s) du \\ & + \overline{\omega} \left(\frac{\partial}{\partial s} \Gamma(t, s) \right) P_{\overline{\omega}}(\Gamma, [t, 1], s). \end{aligned} \quad (42)$$

4 The holonomy of Hopf and bare 2-connections

4.1 Local Hopf 2-connections and their 2-curvature

Let $\mathcal{H} = (\partial: I \rightarrow H, \rho)$ be a crossed module of Hopf algebras. Let M be a manifold.

Definition 55 (Local Hopf 2-connection). *A local Hopf 2-connection is given by a triple (ω, m_1, m_2) of differential forms in M :*

$$\omega \in \Omega^1(M, H), \quad m_1, m_2 \in \Omega^2(M, I).$$

We impose the following compatibility relation (called the vanishing of the fake curvature; [BS05]):

$$\partial(m_1) = d\omega; \quad \partial(m_2) = -\frac{1}{2}\omega \wedge^{[\cdot]} \omega.$$

Let h be a formal parameter. From the triple (ω, m_1, m_2) we can build the pair $(\overline{\omega}, \overline{m})$ of graded forms:

$$\overline{\omega} = h\omega; \quad \overline{m} = hm_1 + h^2m_2.$$

Thus $\overline{\omega}$ takes values in $H[[h]]$, \overline{m} takes values in $I[[h]]$, and the vanishing of the fake curvature means that:

$$\partial(\overline{m}) = \mathcal{F}_{\overline{\omega}}$$

where by definition $\mathcal{F}_{\overline{\omega}} \in \Omega^2(M, H)$ is the graded curvature of $\overline{\omega}$, thus $\mathcal{F}_{\overline{\omega}} = d\overline{\omega} - \frac{1}{2}\overline{\omega} \wedge^{[\cdot]} \overline{\omega} = hd\omega - h^2\frac{1}{2}\omega \wedge^{[\cdot]} \omega$.

This definition makes sense for a crossed module of bare algebras (see 4.3) or of Lie algebras (the latter being the usual setting [SW11, SW13]). For the following definition we restrict ourselves to the Hopf algebra case.

Definition 56 (Primitive Hopf 2-connection). *A local Hopf 2-connection (ω, m_1, m_2) is called:*

- *1-primitive if ω takes values in the vector space of primitive elements of H .*
- *Fully primitive if 1-primitive and both m_1 and m_2 take values in the space of primitive elements of I .*

Definition 57 (2-curvature 3-form). *Given a 1-form ω in a manifold M with values in H and a 2-form m with values in I , we put $\omega \wedge^\triangleright m$, as being the I -valued 3-form in M , such that, for each vector fields X, Y, Z :*

$$(\omega \wedge^\triangleright m)(X, Y, Z) = \omega(X) \triangleright m(Y, Z) + \omega(Y) \triangleright m(Z, X) + \omega(Z) \triangleright m(X, Y).$$

Given a local Hopf 2-connection, described by a triple (ω, m_1, m_2) , its 2-curvature \mathcal{M} is the graded 3-form:

$$\mathcal{M} = d\bar{m} - \bar{\omega} \wedge^\triangleright \bar{m}.$$

Explicitly $\mathcal{M} = hdm_1 + h^2dm_2 - h^2\omega \wedge^\triangleright m_1 - h^3\omega \wedge^\triangleright m_2$.

4.2 The exact two-dimensional holonomy

Let $\mathcal{H} = (\partial: I \rightarrow H, \rho)$ be a Hopf algebra crossed module. Let M be a smooth manifold. Let (ω, m_1, m_2) be a local Hopf 2-connection. We define two (in general) distinct two-dimensional holonomies (exact and fuzzy), for a 2-path. This subsection is devoted to the first. Recall the definition of the crossed module of groups $\mathcal{H}_{\mathfrak{gl}}^* = (\partial: I_{\mathfrak{gl}}^* \rightarrow H_{\mathfrak{gl}}^*, \triangleright)$ and of the larger pre-crossed module of groups $\hat{\mathcal{H}}_{\mathfrak{gl}}^* = (\partial: \hat{I}_{\mathfrak{gl}}^* \rightarrow H_{\mathfrak{gl}}^*, \triangleright)$; see 2.4.4. Recall the constructions of the 2-groupoid $\mathcal{C}^\times(\mathcal{H}_{\mathfrak{gl}}^*)$ and of the larger sesquigroupoid $\hat{\mathcal{C}}^\times(\hat{\mathcal{H}}_{\mathfrak{gl}}^*)$; 2.3.1.

4.2.1 Definition of the exact two-dimensional holonomy

Definition 58 (The exact holonomy). *Resume the notation of 3.2.2 and 3.2.3. Suppose that (ω, m_1, m_2) is 1-primitive, thus $P_{\bar{\omega}}(\Gamma, [t, 0], s)$ is group-like for all $t, s \in [0, 1]$. Consider the differential equation in $I[[h]]$:*

$$\begin{aligned} \frac{d}{ds} Q_{(\bar{\omega}, \bar{m})}(\Gamma, [s, s_0]) &= - \left(\int_0^1 P_{\bar{\omega}}(\Gamma, [0, u], s) \triangleright \bar{m} \left(\frac{\partial}{\partial s} \Gamma(u, s), \frac{\partial}{\partial u} \Gamma(u, s) \right) du \right) Q_{(\bar{\omega}, \bar{m})}(\Gamma, [s, s_0]) \\ Q_{(\bar{\omega}, \bar{m})}(\Gamma, [s_0, s_0]) &= 1_I. \end{aligned} \quad (43)$$

By lemma 47 its solution is unique, explicitly being a path-ordered exponential, see below. Put:

$$Q_{(\bar{\omega}, \bar{m})}(\Gamma) \doteq Q_{(\bar{\omega}, \bar{m})}(\Gamma, [1, 0]) \in I[[h]].$$

Clearly, given $s_0, s, s_1 \in [0, 1]$, $Q_{(\bar{\omega}, \bar{m})}(\Gamma, [s, s_0])$ is invertible, and we have:

$$Q_{(\bar{\omega}, \bar{m})}(\Gamma, [s_1, s_0]) = Q_{(\bar{\omega}, \bar{m})}(\Gamma, [s_1, s]) Q_{(\bar{\omega}, \bar{m})}(\Gamma, [s, s_0]). \quad (44)$$

Lemma 59. *If (ω, m_1, m_2) is fully primitive then $Q_{(\bar{\omega}, \bar{m})}(\Gamma, [s, s_0]) \in I_{\mathfrak{gl}}^*$ is group like for all $s, s_0 \in [0, 1]$.*

Proof: This is entirely analogous to the proof of Lemma 52. Note also the lemma below. \square

Lemma 60. *Let $\mathcal{H} = (\partial: I \rightarrow H, \rho)$ be a crossed module of Hopf algebras. The given action of H on I is such that if $x \in H_{\mathfrak{gl}}^*$ and $a \in \text{Prim}(I)[[h]]$ then*

$$x \triangleright a \in \text{Prim}(I)[[h]].$$

Proof: We have, since I is an H -module coalgebra:

$$\begin{aligned} \Delta(x \triangleright a) &= \sum_{(x)} \sum_{(a)} x' \triangleright a' \otimes x'' \triangleright a'' = x \triangleright a \otimes x \triangleright 1 + x \triangleright 1 \otimes x \triangleright a \\ &= x \triangleright a \otimes \epsilon(x) + \epsilon(x) \otimes x \triangleright a = x \triangleright a \otimes 1 + 1 \otimes x \triangleright a. \end{aligned}$$

\square

Theorem 61 (Exact two-dimensional holonomy). *Suppose that (ω, m_1, m_2) is 1-primitive. The assignments:*

$$\text{for a 1-path } \gamma \text{ in } M : \quad \mathcal{F}_{(\bar{\omega}, \bar{m})}^\times(x \xrightarrow{\gamma} y) = * \xrightarrow{P_{\bar{\omega}}(\gamma)^{-1}} *, \quad (45)$$

which is a morphism of the sesquigroupoid $\hat{\mathcal{C}}^\times(\hat{\mathcal{H}}_{\text{gl}}^*)$, and, for a 2-path in M ,

$$\mathcal{F}_{(\bar{\omega}, \bar{m})}^\times \left(x \begin{array}{c} \xrightarrow{\gamma_2} \\ \uparrow \Gamma \\ \xrightarrow{\gamma_1} \end{array} y \right) = * \begin{array}{c} \xrightarrow{(P_{\bar{\omega}}(\gamma_2))^{-1}} \\ \boxed{\times} \uparrow (Q_{(\bar{\omega}, \bar{m})}(\Gamma))^{-1} \\ \xrightarrow{(P_{\bar{\omega}}(\gamma_1))^{-1}} \end{array} *, \quad (46)$$

a 2-morphism of the sesquigroupoid $\hat{\mathcal{C}}^\times(\hat{\mathcal{H}}_{\text{gl}}^*)$, preserve composition of 1-paths, vertical compositions of 2-paths, whiskerings of 2-paths by 1-paths and vertical reverses or 2-paths. If (ω, m_1, m_2) is fully primitive then the image of $\mathcal{F}_{(\bar{\omega}, \bar{m})}^\times$ lives in $\mathcal{C}^\times(\mathcal{H}_{\text{gl}}^*) \subset \hat{\mathcal{C}}^\times(\hat{\mathcal{H}}_{\text{gl}}^*)$, $\mathcal{F}_{(\bar{\omega}, \bar{m})}^\times$ preserving horizontal compositions and reverses.

Proof: That the functor preserves the composition of 1-paths and 2-paths follows from (36) and (44). That the right-hand-side of 46 is a 2-morphism of the sesquigroupoid $\hat{\mathcal{C}}^\times(\hat{\mathcal{H}}_{\text{gl}}^*)$ follows from Lemma 52 and equations (31), (40) and (43). The first Peiffer law thus imply that, for each $s_0, s \in [0, 1]$:

$$\partial(Q_{(\bar{\omega}, \bar{m})}(\Gamma, [s, s_0])) = P_{\bar{\omega}}(\gamma_s)^{-1} P_{\bar{\omega}}(\gamma_{s_0}), \quad (47)$$

since both of these satisfy differential equation (40), and have the same initial condition, by Lemma 47.

The most difficult bit is the preservation of horizontal composition and reverses, in the fully primitive case. We only prove the preservation of horizontal compositions, since horizontal reverses are dealt with exactly in the same way. We will make use of the previous lemma and Theorem 25. Let Γ_1 and Γ_2 be 2-paths, such that their horizontal composition $\Gamma = \Gamma_1 \Gamma_2$ is well defined. We have:

$$\begin{aligned} \frac{d}{ds} Q_{(\bar{\omega}, \bar{m})}(\Gamma, [s, s_0]) &= - \left(\int_0^1 P_{\bar{\omega}}(\Gamma, [0, u], s) \triangleright \bar{m} \left(\frac{\partial}{\partial s} \Gamma(u, s), \frac{\partial}{\partial u} \Gamma(u, s) \right) du \right) Q_{(\bar{\omega}, \bar{m})}(\Gamma, [s, s_0]) \\ &= - \left(\int_0^1 P_{\bar{\omega}}(\Gamma_1, [0, u], s) \triangleright \bar{m} \left(\frac{\partial}{\partial s} \Gamma_1(u, s), \frac{\partial}{\partial u} \Gamma_1(u, s) \right) du \right) Q_{(\bar{\omega}, \bar{m})}(\Gamma, [s, s_0]) \\ &\quad - \left(P_{\bar{\omega}}(\Gamma_1, [1, 0], s) \triangleright \int_0^1 P_{\bar{\omega}}(\Gamma_2, [0, u], s) \triangleright \bar{m} \left(\frac{\partial}{\partial s} \Gamma_2(u, s), \frac{\partial}{\partial u} \Gamma_2(u, s) \right) du \right) Q_{(\bar{\omega}, \bar{m})}(\Gamma, [s, s_0]). \end{aligned}$$

On the other hand, using the Leibniz law, the second Peiffer relation for group crossed modules, and (47):

$$\begin{aligned} &\frac{d}{ds} \left(Q_{(\bar{\omega}, \bar{m})}(\Gamma_1, [s, s_0]) P_{\bar{\omega}}(\Gamma_1, [1, 0], s_0) \triangleright Q_{(\bar{\omega}, \bar{m})}(\Gamma_2, [s, s_0]) \right) \\ &= - \left(\int_0^1 P_{\bar{\omega}}(\Gamma_1, [0, u], s) \triangleright \bar{m} \left(\frac{\partial}{\partial s} \Gamma_1(u, s), \frac{\partial}{\partial u} \Gamma_1(u, s) \right) du \right) \left(Q_{(\bar{\omega}, \bar{m})}(\Gamma_1, [s, s_0]) P_{\bar{\omega}}(\Gamma_1, [0, 1], 0) \triangleright Q_{(\bar{\omega}, \bar{m})}(\Gamma_2, [s, s_0]) \right) \\ &\quad - Q_{(\bar{\omega}, \bar{m})}(\Gamma_1, [s, s_0]) P_{\bar{\omega}}(\Gamma_1, [0, 1], 0) \triangleright \left(\int_0^1 P_{\bar{\omega}}(\Gamma_2, [0, u], s) \triangleright \bar{m} \left(\frac{\partial}{\partial s} \Gamma_2(u, s), \frac{\partial}{\partial u} \Gamma_2(u, [s, s_0]) \right) du \right) \\ &\quad \quad \quad P_{\bar{\omega}}(\Gamma_1, [0, 1], 0) \triangleright Q_{(\bar{\omega}, \bar{m})}(\Gamma_2, [s, s_0]) \\ &= - \left(\int_0^1 P_{\bar{\omega}}(\Gamma_1, [0, u], s) \triangleright \bar{m} \left(\frac{\partial}{\partial s} \Gamma_1(u, s), \frac{\partial}{\partial u} \Gamma_1(u, s) \right) du \right) \left(Q_{(\bar{\omega}, \bar{m})}(\Gamma_1, [s, s_0]) P_{\bar{\omega}}(\Gamma_1, [0, 1], 0) \triangleright Q_{(\bar{\omega}, \bar{m})}(\Gamma_2, [s, s_0]) \right) \\ &\quad - \left(\partial(Q_{(\bar{\omega}, \bar{m})}(\Gamma_1, [s, s_0])) P_{\bar{\omega}}(\Gamma_1, [0, 1], 0) \right) \triangleright \left(\int_0^1 P_{\bar{\omega}}(\Gamma_2, [0, u], s) \triangleright \bar{m} \left(\frac{\partial}{\partial s} \Gamma_2(u, s), \frac{\partial}{\partial u} \Gamma_2(u, [s, s_0]) \right) du \right) \\ &\quad \quad \quad Q_{(\bar{\omega}, \bar{m})}(\Gamma_1, [s, s_0]) \left(P_{\bar{\omega}}(\Gamma_1, [0, 1], 0) \triangleright Q_{(\bar{\omega}, \bar{m})}(\Gamma_2, [s, s_0]) \right) \\ &= - \left(\int_0^1 P_{\bar{\omega}}(\Gamma_1, [0, u], s) \triangleright \bar{m} \left(\frac{\partial}{\partial s} \Gamma_1(u, s), \frac{\partial}{\partial u} \Gamma_1(u, s) \right) du \right) \left(Q_{(\bar{\omega}, \bar{m})}(\Gamma_1, [s, s_0]) P_{\bar{\omega}}(\Gamma_1, [0, 1], 0) \triangleright Q_{(\bar{\omega}, \bar{m})}(\Gamma_2, [s, s_0]) \right) \\ &\quad - \left(P_{\bar{\omega}}(\Gamma_1, [0, 1], s) \right) \triangleright \left(\int_0^1 P_{\bar{\omega}}(\Gamma_2, [0, u], s) \triangleright \bar{m} \left(\frac{\partial}{\partial s} \Gamma_2(u, s), \frac{\partial}{\partial u} \Gamma_2(u, [s, s_0]) \right) du \right) \\ &\quad \quad \quad Q_{(\bar{\omega}, \bar{m})}(\Gamma_1, [s, s_0]) \left(P_{\bar{\omega}}(\Gamma_1, [0, 1], 0) \triangleright Q_{(\bar{\omega}, \bar{m})}(\Gamma_2, [s, s_0]) \right). \end{aligned}$$

The two differential equations coincide. So do initial values. Therefore we proved that, for each $s_0, s \in [0, 1]$:

$$Q_{(\bar{\omega}, \bar{m})}(\Gamma, [s, s_0]) = Q_{(\bar{\omega}, \bar{m})}(\Gamma_1, [s, s_0]) P_{\bar{\omega}}(\Gamma_1, [0, 1], s_0) \triangleright Q_{(\bar{\omega}, \bar{m})}(\Gamma_2, [s, s_0]). \quad (48)$$

(Note that we used the second Peiffer relation for group crossed modules, thus (48) would not be true if non-fully primitive Hopf 2-connections were used.) Putting $s = 1$ and $s_0 = 0$ and glancing at equations (46) and (25), we see that \mathcal{F} preserves horizontal compositions of 2-paths. Preservation of whiskerings (valid for 1-primitive connections) is dealt with similarly, except that the second Peiffer condition is not needed. \square

4.2.2 The exact two-dimensional holonomy and 2-curvature

Consider a 3-path J connecting the 2-paths Γ_0 and Γ_1 ; Definition 43. Let $\mathcal{H} = (\partial: I \rightarrow H, \rho)$ be a Hopf algebra crossed module. Let M be a smooth manifold, provided with a primitive local Hopf 2-connection (ω, m_1, m_2) . The difference between the exact two-dimensional holonomies $Q_{(\overline{\omega}, \overline{m})}(\Gamma_0)$ and $Q_{(\overline{\omega}, \overline{m})}(\Gamma_1)$ is essentially ruled by the 2-curvature 3-form \mathcal{M} of $(\overline{\omega}, \overline{m})$, Definition 57. In the Lie crossed module case this appeared in [BS05, SW13, FP10]. We essentially use the argument of [BS05], which was generalized to the pre-crossed module case in [FP11b].

Given $t, s, x \in [0, 1]$, put $P_{\overline{\omega}}(J, [t, 1], s, x) = P_{\overline{\omega}}(J_{(s,x)}, [t, 1])$, where $J_{(s,x)}(t) = J(t, s, x)$, a 1-path. Let $[0, 1]^2 = \{(s, x) \in \mathbb{R}^2 : s, x \in [0, 1]\}$. Consider the smooth differential form a in $[0, 1]^2$, with values in I , with:

$$\begin{aligned} a \left(\frac{\partial}{\partial s} \right)_{(x,s)} &= - \int_0^1 P_{\overline{\omega}}(J, [0, u], s, x) \triangleright \overline{m} \left(\frac{\partial}{\partial s} J(u, s, x), \frac{\partial}{\partial u} J(u, s, x) \right) du \\ &= - \int_0^1 P_{\overline{\omega}}(J, [0, u], s, x) \triangleright J^*(\overline{m}) \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial u} \right) du. \\ a \left(\frac{\partial}{\partial x} \right)_{(x,s)} &= - \int_0^1 P_{\overline{\omega}}(J, [0, u], s, x) \triangleright \overline{m} \left(\frac{\partial}{\partial x} J(u, s, x), \frac{\partial}{\partial u} J(u, s, x) \right) du \\ &= - \int_0^1 P_{\overline{\omega}}(J, [0, u], s, x) \triangleright J^*(\overline{m}) \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial u} \right) du. \end{aligned}$$

By using equations (43) and (37), and Leibniz law, its exterior derivative is seen to be such that:

$$\begin{aligned} da \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial s} \right) &= \frac{\partial}{\partial x} a \left(\frac{\partial}{\partial s} \right)_{(x,s)} - \frac{\partial}{\partial s} a \left(\frac{\partial}{\partial x} \right)_{(x,s)} \\ &= \int_0^1 P_{\overline{\omega}}(J, [0, u], s, x) \triangleright \left(J^*(\overline{\omega}) \left(\frac{\partial}{\partial x} \right) \triangleright J^*(\overline{m}) \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial u} \right) - J^*(\overline{\omega}) \left(\frac{\partial}{\partial s} \right) \triangleright J^*(\overline{m}) \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial u} \right) \right) du \\ &\quad - \int_0^1 P_{\overline{\omega}}(J, [0, u], s, x) \triangleright \left(\frac{\partial}{\partial x} J^*(\overline{m}) \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial u} \right) - \frac{\partial}{\partial s} J^*(\overline{m}) \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial u} \right) \right) du \\ &\quad + \int_0^1 \int_0^t P_{\overline{\omega}}(J, [0, u], s, x) J^*(\overline{\omega}) \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial u} \right) P_{\overline{\omega}}(J, [u, 0], s, x) P_{\overline{\omega}}(J, [0, t], s, x) \triangleright J^*(\overline{m}) \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right) du dt \\ &\quad - \int_0^1 \int_0^t P_{\overline{\omega}}(J, [0, u], s, x) J^*(\overline{\omega}) \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial u} \right) P_{\overline{\omega}}(J, [u, 0], s, x) P_{\overline{\omega}}(J, [0, t], s, x) \triangleright J^*(\overline{m}) \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t} \right) du dt \\ &= - \int_0^1 P_{\overline{\omega}}(J, [0, u], s, x) \triangleright J^*(\mathcal{M}) \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial s}, \frac{\partial}{\partial u} \right) du \\ &\quad - \int_0^1 P_{\overline{\omega}}(J, [0, u], s, x) \triangleright \left(\frac{\partial}{\partial u} J^*(\overline{m}) \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial s} \right) - J^*(\overline{\omega}) \left(\frac{\partial}{\partial u} \right) \triangleright J^*(\overline{m}) \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial s} \right) \right) du \\ &\quad + \int_0^1 \int_0^t P_{\overline{\omega}}(J, [0, u], s, x) J^*(\overline{\omega}) \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial u} \right) P_{\overline{\omega}}(J, [u, 0], s, x) P_{\overline{\omega}}(J, [0, t], s, x) \triangleright J^*(\overline{m}) \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right) du dt \\ &\quad - \int_0^1 \int_0^t P_{\overline{\omega}}(J, [0, u], s, x) J^*(\overline{\omega}) \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial u} \right) P_{\overline{\omega}}(J, [u, 0], s, x) P_{\overline{\omega}}(J, [0, t], s, x) \triangleright J^*(\overline{m}) \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t} \right) du dt \end{aligned}$$

Note that, integrating by parts, noting $\frac{\partial}{\partial s} \Gamma_x(0, s) = 0 = \frac{\partial}{\partial s} \Gamma_x(0, s) = 0$, and equation (35), we have:

$$\int_0^1 P_{\overline{\omega}}(J, [0, u], s, x) \triangleright \left(\frac{\partial}{\partial u} J^*(\overline{m}) \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial s} \right) \right) du = - \int_0^1 P_{\overline{\omega}}(J, [0, u], s, x) \triangleright \left(J^*(\overline{\omega}) \left(\frac{\partial}{\partial u} \right) \triangleright J^*(\overline{m}) \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial s} \right) \right) du.$$

This cancels out the second line of the previous formula. Finally:

$$\begin{aligned} [a, a] \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial s} \right) &= \int_0^1 \int_0^1 \left[P_{\overline{\omega}}(J, [0, u], s, x) \triangleright J^*(\overline{m}) \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial u} \right) du, P_{\overline{\omega}}(J, [0, t], s, x) \triangleright J^*(\overline{m}) \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t} \right) \right] du dt \\ &= \int_0^1 \int_0^t \left[P_{\overline{\omega}}(J, [0, u], s, x) \triangleright J^*(\overline{m}) \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial u} \right) du, P_{\overline{\omega}}(J, [0, t], s, x) \triangleright J^*(\overline{m}) \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t} \right) \right] du dt \\ &\quad + \int_0^1 \int_0^u \left[P_{\overline{\omega}}(J, [0, u], s, x) \triangleright J^*(\overline{m}) \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial u} \right) du, P_{\overline{\omega}}(J, [0, t], s, x) \triangleright J^*(\overline{m}) \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t} \right) \right] dt du. \end{aligned}$$

Since $\partial(\overline{m}) = \overline{\omega}$, from the above and (32), we can see that the curvature $da + [a, a]$ of a satisfies:

$$\begin{aligned} (da + [a, a]) \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial s} \right) &= - \int_0^1 P_{\overline{\omega}}(J, [0, u], s, x) \triangleright \left(J^*(\mathcal{M}) \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial s}, \frac{\partial}{\partial u} \right) \right) du \\ &- \int_0^1 \int_0^t \left\langle P_{\overline{\omega}}(J, [0, u], s, x) \triangleright J^*(\overline{m}) \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial u} \right), P_{\overline{\omega}}(J, [0, u], s, x) \triangleright J^*(\overline{m}) \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right) \right\rangle dudt \\ &+ \int_0^1 \int_0^t \left\langle P_{\overline{\omega}}(J, [0, u], s, x) \triangleright J^*(\overline{m}) \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial u} \right), P_{\overline{\omega}}(J, [0, u], s, x) \triangleright J^*(\overline{m}) \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t} \right) \right\rangle dudt. \end{aligned}$$

By using Lemma 54, we can see, given that $Q_{(\overline{\omega}, \overline{m})}(\Gamma, [s, s_0]) = \sum_{n=0}^{+\infty} h^n \underbrace{\oint_{s_0}^s a(\gamma'(v)) * \dots * a(\gamma'(v))}_{n \text{ terms}} dv$:

Theorem 62. *We have, using the notation of Definition 43, thus $J_x(t, s) = \Gamma(t, s, x)$:*

$$\begin{aligned} \frac{d}{dx} Q_{(\overline{\omega}, \overline{m})}(J_x) &= - \int_0^1 \int_0^1 Q_{\omega}(J, [1, s], x) \mathcal{M} \left(\frac{\partial}{\partial x} J(t, s, x), \frac{\partial}{\partial s} J(t, s, x), \frac{\partial}{\partial t} J(t, s, x) \right) Q_{\omega}(J, [s, 0], x) dt ds \\ &+ \int_0^1 \int_0^1 \int_0^t Q_{\omega}(J, [1, s], x) \left\langle P_{\overline{\omega}}(J, [0, u], s, x) \triangleright \overline{m} \left(\frac{\partial}{\partial s} J(u, s, x), \frac{\partial}{\partial u} J(t, s, x) \right), \right. \\ &\quad \left. P_{\overline{\omega}}(J, [0, u], s, x) \triangleright \overline{m} \left(\frac{\partial}{\partial x} J(t, s, x), \frac{\partial}{\partial t} J(t, s, x) \right) \right\rangle Q_{\omega}(J, [s, 0], x) du dt ds \\ &- \int_0^1 \int_0^1 \int_0^t Q_{\omega}(J, [1, s], x) \left\langle P_{\overline{\omega}}(J, [0, u], s, x) \triangleright \overline{m} \left(\frac{\partial}{\partial x} J(u, s, x), \frac{\partial}{\partial u} J(t, s, x) \right), \right. \\ &\quad \left. P_{\overline{\omega}}(J, [0, u], s, x) \triangleright \overline{m} \left(\frac{\partial}{\partial s} J(t, s, x), \frac{\partial}{\partial t} J(t, s, x) \right) \right\rangle Q_{\omega}(J, [s, 0], x) du dt ds \\ &+ \left(\int_0^1 P(J^t, [0, s], x) \triangleright \overline{m} \left(\frac{\partial}{\partial x} J^t(s, x), \frac{\partial}{\partial s} J^t(s, x) \right) du \right) Q_{(\overline{\omega}, \overline{m})}(\Gamma_x) \\ &- Q_{(\overline{\omega}, \overline{m})}(\Gamma_x) \left(\int_0^1 P(J^b, [0, s], x) \triangleright \overline{m} \left(\frac{\partial}{\partial x} J^b(s, x), \frac{\partial}{\partial s} J^b(s, x) \right) du \right). \end{aligned}$$

The tensor $\langle \cdot, \cdot \rangle$ vanishes in the primitive space of $I[[h]]$. Therefore, for fully primitive connections the second and third terms of the previous equation are zero. If we, furthermore, suppose that J is a thin homotopy then all of the remaining terms disappear. Thence we can see that the two-dimensional holonomy of a fully primitive connection is invariant under thin homotopy; c.f. the comments at the end of Lemma 54. Thus:

Theorem 63. *Suppose that $(\overline{\omega}, \overline{m})$ is fully primitive. The assignments $\mathcal{F}_{(\overline{\omega}, \overline{m})}^{\times}$, defined in (45) and (46), descend to a functor from the fundamental thin 2-groupoid of M (see Remark 42) into the 2-groupoid $\mathcal{C}^{\times}(\mathcal{H}_{\text{gl}}^*)$.*

Also:

Theorem 64. *Let (ω, m_1, m_2) be a fully primitive \mathcal{H} -valued Hopf 2-connection in a manifold M ; where $\mathcal{H} = (\partial: I \rightarrow H, \rho)$. Suppose that the 2-curvature 3-form $\mathcal{M} = d\overline{m} - \overline{\omega} \wedge^{\triangleright} \overline{m}$ vanishes (Definition 57), explicitly $dm_1 = 0$, $dm_2 - \omega \wedge^{\triangleright} m_1 = 0$ and $\omega \wedge^{\triangleright} m_2 = 0$. The two-dimensional holonomy $\mathcal{F}_{(\overline{\omega}, \overline{m})}^{\times}(\Gamma) \in I_{\text{gl}}^*$ of a 2-path $\Gamma \in \mathcal{P}^2(M)$ depends only on the homotopy class of $\Gamma: [0, 1]^2 \rightarrow M$, relative to the boundary.*

4.3 Bare 2-connections and their holonomy

Consider a unital pre-crossed module $\hat{A} = (\partial: A \rightarrow B, \triangleright, \triangleleft)$, of bare algebras. The following parallels the discussion in subsection 4.1. A local bare 2-connection, on a manifold M , yields a two-dimensional holonomy, living in the sesquigroupoid $\hat{\mathcal{C}}^+(\hat{A})$; see 2.3.2. This sesquigroupoid is a 2-groupoid if \hat{A} is a crossed module.

Definition 65 (Bare 2-connection). *A (local) bare 2-connection is given by a triple (ω, m_1, m_2) of differential forms in M , namely $\omega \in \Omega^1(M, B)$ and $m_1, m_2 \in \Omega^2(M, A)$, such that the vanishing of the fake curvature condition holds: $\partial(m_1) = d\omega$ and $\partial(m_2) = -\frac{1}{2}\omega \wedge^{[\cdot]} \omega$. Given a formal parameter h we can build the pair $(\overline{\omega}, \overline{m})$ of graded forms: $\overline{\omega} = h\omega$ and $\overline{m} = hm_1 + h^2m_2$. Therefore we have $\partial(\overline{m}) = \mathcal{F}_{\overline{\omega}}$ where $\mathcal{F}_{\overline{\omega}} \in \Omega^2(M, B[[h]])$ is the graded curvature of $\overline{\omega}$, thus: $\mathcal{F}_{\overline{\omega}} = d\overline{\omega} - \frac{1}{2}\overline{\omega} \wedge^{[\cdot]} \overline{\omega} = hd\omega - h^2\frac{1}{2}\omega \wedge^{[\cdot]} \omega$.*

Definition 66 (Curvature of a bare 2-connection). *The bare 2-curvature 3-form of the bare 2-connection (ω, m_1, m_2) is the graded 3-form $\mathcal{M} \in \Omega^3(M, A[[h]])$, such that:*

$$\mathcal{M} = d\bar{m} - \bar{\omega} \wedge^\flat \bar{m},$$

where $b \triangleright a \doteq b \triangleright a - a \triangleleft b$, for $a \in A$ and $b \in B$. If X, Y, Z are vector fields in M , then for example:

$$\begin{aligned} (\omega \wedge^\flat m_1)(X, Y, Z) &= \omega(X) \triangleright m(Y, Z) + \omega(Y) \triangleright m(Z, X) + \omega(Z) \triangleright m(X, Y) \\ &\quad - m(Y, Z) \triangleleft \omega(X) + m(Z, X) \triangleleft \omega(Y) + m(X, Y) \triangleleft \omega(Z). \end{aligned}$$

4.3.1 The two dimensional holonomy of a bare 2-connection

We resume the notation of 3.2.3. Let M be a manifold, $\hat{A} = (\partial: A \rightarrow B, \triangleright, \triangleleft)$ a pre-crossed module of bare algebras, and (ω, m_1, m_2) a bare 2-connection, seen as a pair $(\bar{\omega}, \bar{m})$ of graded forms. Looking at (40), and

considering a 2-path $x \begin{array}{c} \xrightarrow{\gamma_1} \\ \uparrow \Gamma \\ \xrightarrow{\gamma_0} \end{array} y$ in M , we put:

$$R_{(\bar{\omega}, \bar{m})}(\Gamma, [s, s_0]) = - \int_{s_0}^s \int_0^1 P_{\bar{\omega}}(\Gamma, [0, u], v) \triangleright \bar{m} \left(\frac{\partial}{\partial v} \Gamma(u, v), \frac{\partial}{\partial u} \Gamma(u, v) \right) \triangleleft P_{\bar{\omega}}(\Gamma, [u, 1], v) du dv \in A. \quad (49)$$

Also put $R_{(\bar{\omega}, \bar{m})}(\Gamma) \doteq R_{(\bar{\omega}, \bar{m})}(\Gamma, [1, 0])$.

Theorem 67. *The assignment:*

$$\text{for a 1-path } \gamma \text{ in } M : \quad \mathcal{F}_{(\bar{\omega}, \bar{m})}^+(x \xrightarrow{\gamma} y) = * \xrightarrow{P_{\bar{\omega}}(\gamma)^{-1}} *, \quad (50)$$

which is a morphism of the sesquigroupoid $\hat{\mathcal{C}}^+(\hat{A})$, and, for a 2-path in M ,

$$\mathcal{F}_{(\bar{\omega}, \bar{m})}^+ \left(x \begin{array}{c} \xrightarrow{\gamma_2} \\ \uparrow \Gamma \\ \xrightarrow{\gamma_1} \end{array} y \right) = * \begin{array}{c} \xrightarrow{(P_{\bar{\omega}}(\gamma_2))^{-1}} \\ \boxed{+} \uparrow R_{(\bar{\omega}, \bar{m})}(\Gamma) \\ \xrightarrow{(P_{\bar{\omega}}(\gamma_1))^{-1}} \end{array} *, \quad (51)$$

which is a 2-morphism of the sesquigroupoid $\hat{\mathcal{C}}^+(\hat{A})$, preserves composition of 1-paths, vertical compositions of 2-paths, whiskerings of 2-paths by 1-paths, and vertical reverses or 2-paths. If furthermore \hat{A} is a crossed module of bare algebras, then $\mathcal{F}_{(\bar{\omega}, \bar{m})}^+$ also preserves horizontal compositions and horizontal reverses of 2-paths.

Proof: That the image of $\mathcal{F}_{(\bar{\omega}, \bar{m})}^+$ is in the sesquigroupoid $\hat{\mathcal{C}}^+(\hat{A})$ follows from (49), (40), (36) and the definition of crossed modules of bare algebras; see 2.2.1. These imply that, for each $s, s_0 \in [0, 1]$, and each 2-path Γ :

$$P_{\bar{\omega}}(\Gamma, [0, u], s) = P_{\bar{\omega}}(\Gamma, [0, u], s_0) + \partial (R_{(\bar{\omega}, \bar{m})}(\Gamma, [s, s_0])). \quad (52)$$

The most difficult bit is the preservation of horizontal compositions and reverses in the crossed module case. We deal only with horizontal compositions, since horizontal reverses are dealt with similarly. Consider two 2-paths Γ_1 and Γ_2 such that $\Gamma = \Gamma_1 \Gamma_2$, their horizontal composition, is well defined. We have:

$$\begin{aligned} R_{(\bar{\omega}, \bar{m})}(\Gamma, [s, s_0]) &= - \int_{s_0}^s \int_0^1 P_{\bar{\omega}}(\Gamma, [0, u], v) \triangleright \bar{m} \left(\frac{\partial}{\partial v} \Gamma(u, v), \frac{\partial}{\partial u} \Gamma(u, v) \right) \triangleleft P_{\bar{\omega}}(\Gamma, [u, 1], v) du dv \\ &= - \int_{s_0}^s \int_0^1 P_{\bar{\omega}}(\Gamma_1, [0, 1], v) P_{\bar{\omega}}(\Gamma_2, [0, u], v) \triangleright \bar{m} \left(\frac{\partial}{\partial v} \Gamma_2(u, v), \frac{\partial}{\partial u} \Gamma_2(u, v) \right) \triangleleft P_{\bar{\omega}}(\Gamma_2, [u, 1], v) du dv \\ &\quad - \int_{s_0}^s \int_0^1 P_{\bar{\omega}}(\Gamma_1, [0, u], v) \triangleright \bar{m} \left(\frac{\partial}{\partial v} \Gamma_2(u, v), \frac{\partial}{\partial u} \Gamma_2(u, v) \right) \triangleleft P_{\bar{\omega}}(\Gamma_1, [u, 1], v) P_{\bar{\omega}}(\Gamma_2, [0, 1], v) du dv. \end{aligned}$$

Now use (52) in $P_{\bar{\omega}}(\Gamma_2, [0, 1], v)$ (end of the bottom line) and in $P_{\bar{\omega}}(\Gamma_1, [0, 1], v)$ (beginning of the top line):

$$R_{(\bar{\omega}, \bar{m})}(\Gamma, [s, s_0]) = R_{(\bar{\omega}, \bar{m})}(\Gamma_1, [s, s_0]) \triangleleft P_{\bar{\omega}}(\Gamma_2, [0, 1], s_0) + P_{\bar{\omega}}(\Gamma_1, [0, u], s_0) \triangleright R_{(\bar{\omega}, \bar{m})}(\Gamma_2, [s, s_0]) + A,$$

where, using (49), and ulteriorly the second Peiffer relation for crossed modules of bare algebras:

$$\begin{aligned}
A &= \int_{s_0}^s \int_0^1 \int_{s_0}^v \int_0^1 P_{\overline{\omega}}(\Gamma_1, [0, u], v) \triangleright \overline{m} \left(\frac{\partial}{\partial v} \Gamma_1(u, v), \frac{\partial}{\partial u} \Gamma_1(u, v) \right) \triangleleft P_{\overline{\omega}}(\Gamma_1, [u, 1], v) \\
&\quad \partial \left(P_{\overline{\omega}}(\Gamma_2, [0, u'], v') \triangleright \overline{m} \left(\frac{\partial}{\partial v'} \Gamma_2(u', v'), \frac{\partial}{\partial u'} \Gamma_2(u', v') \right) \triangleleft P_{\overline{\omega}}(\Gamma_2, [u', 1], v') \right) du' dv' du dv \\
&+ \int_{s_0}^s \int_0^1 \int_{s_0}^v \int_0^1 \partial \left(P_{\overline{\omega}}(\Gamma_1, [0, u'], v') \triangleright \overline{m} \left(\frac{\partial}{\partial v'} \Gamma_1(u', v'), \frac{\partial}{\partial u'} \Gamma_1(u', v') \right) \triangleleft P_{\overline{\omega}}(\Gamma_1, [u', 1], v') \right) \\
&\quad P_{\overline{\omega}}(\Gamma_2, [0, u], v) \triangleright \overline{m} \left(\frac{\partial}{\partial v} \Gamma_2(u, v), \frac{\partial}{\partial u} \Gamma_2(u, v) \right) \triangleleft P_{\overline{\omega}}(\Gamma_2, [u, 1], v) du' dv' du dv \\
&= \int_{s_0}^s \int_0^1 \int_{s_0}^v \int_0^1 P_{\overline{\omega}}(\Gamma_1, [0, u], v) \triangleright \overline{m} \left(\frac{\partial}{\partial v} \Gamma_1(u, v), \frac{\partial}{\partial u} \Gamma_1(u, v) \right) \triangleleft P_{\overline{\omega}}(\Gamma_1, [u, 1], v) \\
&\quad P_{\overline{\omega}}(\Gamma_2, [0, u'], v') \triangleright \overline{m} \left(\frac{\partial}{\partial v'} \Gamma_2(u', v'), \frac{\partial}{\partial u'} \Gamma_2(u', v') \right) \triangleleft P_{\overline{\omega}}(\Gamma_2, [u', 1], v') du' dv' du dv \\
&+ \int_{s_0}^s \int_0^1 \int_{s_0}^v \int_0^1 P_{\overline{\omega}}(\Gamma_1, [0, u'], v') \triangleright \overline{m} \left(\frac{\partial}{\partial v'} \Gamma_1(u', v'), \frac{\partial}{\partial u'} \Gamma_1(u', v') \right) \triangleleft P_{\overline{\omega}}(\Gamma_1, [u', 1], v') \\
&\quad P_{\overline{\omega}}(\Gamma_2, [0, u], v) \triangleright \overline{m} \left(\frac{\partial}{\partial v} \Gamma_2(u, v), \frac{\partial}{\partial u} \Gamma_2(u, v) \right) \triangleleft P_{\overline{\omega}}(\Gamma_2, [u, 1], v) du' dv' du dv \\
&= \int_{s_0}^s \int_0^1 \int_{s_0}^s \int_0^1 P_{\overline{\omega}}(\Gamma_1, [0, u'], v') \triangleright \overline{m} \left(\frac{\partial}{\partial v'} \Gamma_1(u', v'), \frac{\partial}{\partial u'} \Gamma_1(u', v') \right) \triangleleft P_{\overline{\omega}}(\Gamma_1, [u', 1], v') \\
&\quad P_{\overline{\omega}}(\Gamma_2, [0, u], v) \triangleright \overline{m} \left(\frac{\partial}{\partial v} \Gamma_2(u, v), \frac{\partial}{\partial u} \Gamma_2(u, v) \right) \triangleleft P_{\overline{\omega}}(\Gamma_2, [u, 1], v) du dv du' dv' \\
&= R_{(\overline{\omega}, \overline{m})}(\Gamma_1, [s, s_0]) R_{(\overline{\omega}, \overline{m})}(\Gamma_2, [s, s_0]).
\end{aligned}$$

Therefore the following holds in the crossed module case (but not-necessarily in the pre-crossed module case):

$$\begin{aligned}
R_{(\overline{\omega}, \overline{m})}((\Gamma_1 \Gamma_2), [s, s_0]) &= R_{(\overline{\omega}, \overline{m})}(\Gamma_1, [s, s_0]) \triangleleft P_{\overline{\omega}}(\Gamma_2, [0, 1], s_0) \\
&\quad + P_{\overline{\omega}}(\Gamma_1, [0, u], s_0) \triangleright R_{(\overline{\omega}, \overline{m})}(\Gamma_2, [s, s_0]) + R_{(\overline{\omega}, \overline{m})}(\Gamma_1, [s, s_0]) R_{(\overline{\omega}, \overline{m})}(\Gamma_2, [s, s_0]). \quad (53)
\end{aligned}$$

Putting $s_0 = 0, s_1 = 1$ and looking at (28), we see that $\mathcal{F}_{(\overline{\omega}, \overline{m})}^+$ preserves horizontal composition of 2-paths. Horizontal reverses are dealt with similarly. \square

4.3.2 Bare 2-curvature and two dimensional holonomy

Recall Definition (43). Consider a 3-path J connecting the 2-paths Γ_0 and Γ_1 . Consider a unital pre-crossed module $\hat{A} = (\partial: A \rightarrow B, \triangleright, \triangleleft)$, of bare algebras. Consider a bare 2-connection (ω, m_1, m_2) in a manifold M . The difference between the bare two-dimensional holonomies $R_{(\overline{\omega}, \overline{m})}(\Gamma_0)$ and $R_{(\overline{\omega}, \overline{m})}(\Gamma_1)$ is determined by the 2-curvature 3-form \mathcal{M} of $(\overline{\omega}, \overline{m})$, Definition 57. In the chain-complex case this appeared in [CFM12a], and we use essentially the same argument.

Given $x \in [0, 1]$, we have a 2-path Γ_x , where $\Gamma_x(t, s) = J(t, s, x)$. For $t, s, x \in [0, 1]$, put $P_{\overline{\omega}}(J, [t, 1], s, x) = P_{\overline{\omega}}(J_{(s,x)}, [t, 1])$, where $J_{(s,x)}(t) = J(t, s, x)$. We have (where we use Leibniz law and equations (41), (42)):

$$\begin{aligned}
\frac{d}{dx} R_{(\overline{\omega}, \overline{m})}(\Gamma_x) &= -\frac{d}{dx} \int_0^1 \int_0^1 P_{\overline{\omega}}(\Gamma_x, [0, u], v) \triangleright \overline{m} \left(\frac{\partial}{\partial v} \Gamma_x(u, v), \frac{\partial}{\partial u} \Gamma_x(u, v) \right) \triangleleft P_{\overline{\omega}}(\Gamma_x, [u, 1], v) du dv \\
&= -\int_0^1 \left(\frac{d}{dx} \int_0^1 P_{\overline{\omega}}(\Gamma_x, [0, u], v) \right) \triangleright \overline{m} \left(\frac{\partial}{\partial v} \Gamma_x(u, v), \frac{\partial}{\partial u} \Gamma_x(u, v) \right) \triangleleft P_{\overline{\omega}}(\Gamma_x, [u, 1], v) du dv \\
&\quad - \int_0^1 \int_0^1 P_{\overline{\omega}}(\Gamma_x, [0, u], v) \triangleright J^*(\overline{m}) \left(\frac{\partial}{\partial v}, \frac{\partial}{\partial u} \right) \triangleleft \left(\frac{d}{dx} P_{\overline{\omega}}(\Gamma_x, [u, 1], v) \right) du dv \\
&\quad - \int_0^1 \int_0^1 P_{\overline{\omega}}(\Gamma_x, [0, u], v) \triangleright \left(\frac{d}{dx} \overline{m} \left(\frac{\partial}{\partial v} \Gamma_x(u, v), \frac{\partial}{\partial u} \Gamma_x(u, v) \right) \right) \triangleleft P_{\overline{\omega}}(\Gamma_x, [u, 1], v) du dv \\
&= \int_0^1 \int_0^1 \int_0^u P_{\overline{\omega}}(\Gamma_x, [0, u'], v) \mathcal{F}_{\overline{\omega}} \left(\frac{\partial}{\partial x} \Gamma_x(u', v), \frac{\partial}{\partial u'} \Gamma_x(u', v) \right)
\end{aligned}$$

$$\begin{aligned}
& P_{\bar{\omega}}(\Gamma_x, [u', u], v) \succ J^*(\bar{m}) \left(\frac{\partial}{\partial v}, \frac{\partial}{\partial u} \right) \prec P_{\bar{\omega}}(\Gamma_x, [u, 1], v) du' du dv \\
& + \int_0^1 \int_0^1 P_{\bar{\omega}}(\Gamma, [0, u], v) \bar{\omega} \left(\frac{\partial}{\partial x} \Gamma_x(u, v) \right) \succ J^*(\bar{m}) \left(\frac{\partial}{\partial v}, \frac{\partial}{\partial u} \right) \prec P_{\bar{\omega}}(\Gamma_x, [u, 1], v) du dv \\
& + \int_0^1 \int_0^1 \int_u^1 P_{\bar{\omega}}(\Gamma_x, [0, u], v) \succ J^*(\bar{m}) \left(\frac{\partial}{\partial v}, \frac{\partial}{\partial u} \right) \prec P_{\bar{\omega}}(\Gamma, [u, u'], v) \\
& \quad \mathcal{F}_{\bar{\omega}} \left(\frac{\partial}{\partial x} \Gamma_x(u, v), \frac{\partial}{\partial u'} \Gamma_x(u, v) \right) P_{\bar{\omega}}(\Gamma_x, [u', 1], v) du' du dv \\
& - \int_0^1 \int_0^1 P_{\bar{\omega}}(\Gamma_x, [0, u], v) \succ J^*(\bar{m}) \left(\frac{\partial}{\partial v}, \frac{\partial}{\partial u} \right) \prec \bar{\omega} \left(\frac{\partial}{\partial x} \Gamma_x(u, v) \right) P_{\bar{\omega}}(\Gamma, [u, 1], v) \cdot du dv \\
& - \int_0^1 \int_0^1 P_{\bar{\omega}}(\Gamma_x, [0, u], v) \succ J^*(d\bar{m}) \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial v}, \frac{\partial}{\partial u} \right) \prec P_{\bar{\omega}}(\Gamma_x, [u, 1], v) du dv \\
& + \int_0^1 \int_0^1 P_{\bar{\omega}}(\Gamma_x, [0, u], v) \succ \left(\frac{\partial}{\partial v} J^*(\bar{m}) \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial x} \right) + \frac{\partial}{\partial u} J^*(\bar{m}) \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial v} \right) \right) \prec P_{\bar{\omega}}(\Gamma_x, [u, 1], v) du dv.
\end{aligned}$$

Now look at the last term. Integrating by parts, and using (41) and (42), we have:

$$\begin{aligned}
& \int_0^1 \int_0^1 P_{\bar{\omega}}(\Gamma_x, [0, u], v) \succ \frac{\partial}{\partial v} J^*(\bar{m}) \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial x} \right) \prec P_{\bar{\omega}}(\Gamma_x, [u, 1], v) du dv \\
& = - \int_0^1 \int_0^1 \left(\frac{\partial}{\partial v} P_{\bar{\omega}}(\Gamma_x, [0, u], v) \right) \succ J^*(\bar{m}) \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial x} \right) \prec P_{\bar{\omega}}(\Gamma_x, [u, 1], v) du dv \\
& \quad - \int_0^1 \int_0^1 P_{\bar{\omega}}(\Gamma_x, [0, u], v) \succ J^*(\bar{m}) \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial x} \right) \prec \left(\frac{\partial}{\partial v} P_{\bar{\omega}}(\Gamma_x, [u, 1], v) \right) du dv \\
& \quad + \int_0^1 P_{\bar{\omega}}(\Gamma_x, [0, u], 1) \succ J^*(\bar{m}) \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial x} \right) \prec P_{\bar{\omega}}(\Gamma_x, [u, 1], 1) du \\
& \quad - \int_0^1 P_{\bar{\omega}}(\Gamma_x, [0, u], 0) \succ J^*(\bar{m}) \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial x} \right) \prec P_{\bar{\omega}}(\Gamma_x, [u, 1], 0) du \\
& = \int_0^1 \int_0^1 \int_0^u P_{\bar{\omega}}(\Gamma_x, [0, u'], v) \mathcal{F}_{\bar{\omega}} \left(\frac{\partial}{\partial v} \Gamma_x(u', v), \frac{\partial}{\partial u'} \Gamma_x(u', v) \right) \\
& \quad P_{\bar{\omega}}(\Gamma, [u', u], v) \succ J^*(\bar{m}) \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial x} \right) \prec P_{\bar{\omega}}(\Gamma_x, [u, 1], v) du' du dv \\
& \quad + \int_0^1 \int_0^1 P_{\bar{\omega}}(\Gamma, [0, u], v) \bar{\omega} \left(\frac{\partial}{\partial v} \Gamma_x(u, v) \right) \succ J^*(\bar{m}) \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial x} \right) \prec P_{\bar{\omega}}(\Gamma_x, [u, 1], v) du dv \\
& \quad + \int_0^1 \int_0^1 \int_u^1 P_{\bar{\omega}}(\Gamma_x, [0, u], v) \succ J^*(\bar{m}) \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial x} \right) \prec P_{\bar{\omega}}(\Gamma, [u, u'], v) \\
& \quad \quad \mathcal{F}_{\bar{\omega}} \left(\frac{\partial}{\partial v} \Gamma_x(u, v), \frac{\partial}{\partial u'} \Gamma_x(u, v) \right) P_{\bar{\omega}}(\Gamma_x, [u', 1], v) du' du dv \\
& \quad - \int_0^1 \int_0^1 P_{\bar{\omega}}(\Gamma_x, [0, u], v) \succ J^*(\bar{m}) \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial x} \right) \prec \bar{\omega} \left(\frac{\partial}{\partial v} \Gamma_x(u, v) \right) P_{\bar{\omega}}(\Gamma, [u, 1], v) \cdot du dv \\
& \quad + \int_0^1 P_{\bar{\omega}}(\Gamma_x, [0, u], 1) \succ J^*(\bar{m}) \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial x} \right) \prec P_{\bar{\omega}}(\Gamma_x, [u, 1], 1) du \\
& \quad - \int_0^1 P_{\bar{\omega}}(\Gamma_x, [0, u], 0) \succ J^*(\bar{m}) \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial x} \right) \prec P_{\bar{\omega}}(\Gamma_x, [u, 1], 0) du.
\end{aligned}$$

Analogously, integrating by parts, using (35), and noting that $J(0, s, x)$ and $J(1, s, x)$ each are constant:

$$\begin{aligned}
& \int_0^1 \int_0^1 P_{\bar{\omega}}(\Gamma_x, [0, u], v) \succ \frac{\partial}{\partial u} J^*(\bar{m}) \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial v} \right) \prec P_{\bar{\omega}}(\Gamma_x, [u, 1], v) du dv \\
& = \int_0^1 \int_0^1 P_{\bar{\omega}}(\Gamma_x, [0, u], v) \bar{\omega} \left(\frac{\partial}{\partial u} \Gamma_x(u, v) \right) \succ J^*(\bar{m}) \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial v} \right) \prec P_{\bar{\omega}}(\Gamma_x, [u, 1], v) du dv
\end{aligned}$$

$$- \int_0^1 \int_0^1 P_{\bar{\omega}}(\Gamma_x, [0, u], v) \succ J^*(\bar{m}) \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial v} \right) \ll \bar{\omega} \left(\frac{\partial}{\partial u} \Gamma_x(u, v) \right) P_{\bar{\omega}}(\Gamma_x, [u, 1], v) du dv.$$

Putting everything together we see:

$$\begin{aligned} \frac{d}{dx} R_{(\bar{\omega}, \bar{m})}(\Gamma_x) &= A - \int_0^1 \int_0^1 P_{\bar{\omega}}(\Gamma_x, [0, u], v) \succ J^*(\mathcal{M}) \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial v}, \frac{\partial}{\partial u} \right) \ll P_{\bar{\omega}}(\Gamma_x, [u, 1], v) du dv \\ &\quad + \int_0^1 P_{\bar{\omega}}(\Gamma_x, [0, u], 1) \succ J^*(\bar{m}) \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial x} \right) \ll P_{\bar{\omega}}(\Gamma_x, [u, 1], 1) du \\ &\quad - \int_0^1 P_{\bar{\omega}}(\Gamma_x, [0, u], 0) \succ J^*(\bar{m}) \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial x} \right) \ll P_{\bar{\omega}}(\Gamma_x, [u, 1], 0) du, \end{aligned}$$

where:

$$\begin{aligned} A &= \int_0^1 \int_0^1 \int_0^u P_{\bar{\omega}}(\Gamma_x, [0, u'], v) \mathcal{F}_{\bar{\omega}} \left(\frac{\partial}{\partial x} \Gamma_x(u', v), \frac{\partial}{\partial u'} \Gamma_x(u', v) \right) \\ &\quad P_{\bar{\omega}}(\Gamma_x, [u', u], v) \succ J^*(\bar{m}) \left(\frac{\partial}{\partial v}, \frac{\partial}{\partial u} \right) \ll P_{\bar{\omega}}(\Gamma_x, [u, 1], v) du' du dv \\ &\quad + \int_0^1 \int_0^1 \int_u^1 P_{\bar{\omega}}(\Gamma_x, [0, u], v) \succ J^*(\bar{m}) \left(\frac{\partial}{\partial v}, \frac{\partial}{\partial u} \right) \ll P_{\bar{\omega}}(\Gamma, [u, u'], v) \\ &\quad \mathcal{F}_{\bar{\omega}} \left(\frac{\partial}{\partial x} \Gamma_x(u, v), \frac{\partial}{\partial u'} \Gamma_x(u, v) \right) P_{\bar{\omega}}(\Gamma_x, [u', 1], v) du' du dv \\ &\quad + \int_0^1 \int_0^1 \int_0^u P_{\bar{\omega}}(\Gamma_x, [0, u'], v) \mathcal{F}_{\bar{\omega}} \left(\frac{\partial}{\partial v} \Gamma_x(u', v), \frac{\partial}{\partial u'} \Gamma_x(u', v) \right) \\ &\quad P_{\bar{\omega}}(\Gamma, [u', u], v) \succ J^*(\bar{m}) \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial x} \right) \ll P_{\bar{\omega}}(\Gamma_x, [u, 1], v) du' du dv \\ &\quad + \int_0^1 \int_0^1 \int_u^1 P_{\bar{\omega}}(\Gamma_x, [0, u], v) \succ J^*(\bar{m}) \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial x} \right) \ll P_{\bar{\omega}}(\Gamma, [u, u'], v) \\ &\quad \mathcal{F}_{\bar{\omega}} \left(\frac{\partial}{\partial v} \Gamma_x(u, v), \frac{\partial}{\partial u'} \Gamma_x(u, v) \right) P_{\bar{\omega}}(\Gamma_x, [u', 1], v) du' du dv = B + C + D + E. \end{aligned}$$

Swapping orders of integration we have:

$$\begin{aligned} B &= \int_0^1 \int_0^1 \int_{u'}^1 P_{\bar{\omega}}(\Gamma_x, [0, u'], v) \mathcal{F}_{\bar{\omega}} \left(\frac{\partial}{\partial x} \Gamma_x(u', v), \frac{\partial}{\partial u'} \Gamma_x(u', v) \right) \\ &\quad P_{\bar{\omega}}(\Gamma_x, [u', u], v) \succ J^*(\bar{m}) \left(\frac{\partial}{\partial v}, \frac{\partial}{\partial u} \right) \ll P_{\bar{\omega}}(\Gamma_x, [u, 1], v) du du' dv. \end{aligned}$$

Let us now use the fact that $\partial(\bar{m}) = \bar{\omega}$, together with equation (19), to see that:

$$\begin{aligned} B + E &= \int_0^1 \int_0^1 \int_{u'}^1 \left\{ P_{\bar{\omega}}(\Gamma_x, [0, u'], v) \bar{m} \left(\frac{\partial}{\partial x} \Gamma_x(u', v), \frac{\partial}{\partial u'} \Gamma_x(u', v) \right), \right. \\ &\quad \left. P_{\bar{\omega}}(\Gamma_x, [u', u], v) \succ J^*(\bar{m}) \left(\frac{\partial}{\partial v}, \frac{\partial}{\partial u} \right) \ll P_{\bar{\omega}}(\Gamma_x, [u, 1], v) \right\} du du' dv. \end{aligned}$$

We analogously reach:

$$\begin{aligned} C + D &= \int_0^1 \int_0^1 \int_0^u \left\{ P_{\bar{\omega}}(\Gamma_x, [0, u'], v) \bar{m} \left(\frac{\partial}{\partial v} \Gamma_x(u', v), \frac{\partial}{\partial u'} \Gamma_x(u', v) \right), \right. \\ &\quad \left. P_{\bar{\omega}}(\Gamma, [u', u], v) \succ J^*(\bar{m}) \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial x} \right) \ll P_{\bar{\omega}}(\Gamma_x, [u, 1], v) \right\} du' du dv. \end{aligned}$$

Theorem 68. *In the conditions stated in the beginning of 4.3.2 we have:*

$$\frac{d}{dx} R_{(\bar{\omega}, \bar{m})}(\Gamma_x)$$

$$\begin{aligned}
&= - \int_0^1 \int_0^1 P_{\overline{\omega}}(\Gamma_x, [0, u], v) \triangleright \mathcal{M} \left(\frac{\partial}{\partial x} J(u, v, x), \frac{\partial}{\partial v} J(u, v, x), \frac{\partial}{\partial u} J(u, v, x) \right) \triangleleft P_{\overline{\omega}}(\Gamma_x, [u, 1], v) du dv \\
&+ \int_0^1 P_{\overline{\omega}}(\Gamma_x, [0, u], 1) \triangleright \overline{m} \left(\frac{\partial}{\partial u} J(u, 1, x), \frac{\partial}{\partial x} J(u, 1, x) \right) \triangleleft P_{\overline{\omega}}(\Gamma_x, [u, 1], 1) du \\
&- \int_0^1 P_{\overline{\omega}}(\Gamma_x, [0, u], 0) \triangleright \overline{m} \left(\frac{\partial}{\partial u} J(u, 0, x), \frac{\partial}{\partial x} J(u, 0, x) \right) \triangleleft P_{\overline{\omega}}(\Gamma_x, [u, 1], 0) du \\
&+ \int_0^1 \int_0^1 \int_{u'}^1 \left\{ P_{\overline{\omega}}(\Gamma_x, [0, u'], v) \overline{m} \left(\frac{\partial}{\partial x} \Gamma_x(u', v), \frac{\partial}{\partial u'} \Gamma_x(u', v) \right), \right. \\
&\quad \left. P_{\overline{\omega}}(\Gamma_x, [u', u], v) \triangleright J^*(\overline{m}) \left(\frac{\partial}{\partial v}, \frac{\partial}{\partial u} \right) \triangleleft P_{\overline{\omega}}(\Gamma_x, [u, 1], v) \right\} du du' dv \\
&+ \int_0^1 \int_0^1 \int_0^u \left\{ P_{\overline{\omega}}(\Gamma_x, [0, u'], v) \overline{m} \left(\frac{\partial}{\partial v} \Gamma_x(u', v), \frac{\partial}{\partial u'} \Gamma_x(u', v) \right), \right. \\
&\quad \left. P_{\overline{\omega}}(\Gamma_x, [u', u], v) \triangleright J^*(\overline{m}) \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial x} \right) \triangleleft P_{\overline{\omega}}(\Gamma_x, [u, 1], v) \right\} du' du dv.
\end{aligned}$$

Remark 69. In a crossed module of bare algebras the tensor $\{, \}$ of (19) vanishes. If we furthermore suppose that J is a thin homotopy then all of the remaining terms of the previous equation vanish, thence we can see that the two-dimensional holonomy of a bare 2-connection with values in a crossed module of bare algebras is invariant under thin homotopy. Also in the crossed module case, the two-dimensional holonomy $\mathcal{F}_{(\overline{\omega}, \overline{m})}^+(\Gamma)$ of a bare 2-connection with zero 2-curvature 3-tensor depends only on the homotopy class of the 2-path $\Gamma: [0, 1]^2 \rightarrow M$, relative to the boundary.

From this remark it follows that:

Corollary 70. Suppose that $\hat{A} = (\partial: A \rightarrow B, \triangleright, \triangleleft)$ is a crossed module of bare algebras. The assignment $\mathcal{F}_{(\overline{\omega}, \overline{m})}^+$ defined in (50) and (51) descends to a 2-functor from the fundamental thin 2-groupoid of M (see remark 42) into the sesquigroupoid $\hat{\mathcal{C}}^+(\hat{A})$, which in this case is a 2-groupoid, since \hat{A} is a crossed module.

4.4 The blur and fuzzy holonomies of a Hopf 2-connection

Let $\mathcal{H} = (\partial: I \rightarrow H, \rho)$ be a crossed module of Hopf algebras. As explained in 2.5.6, the notation and nomenclature of which we freely use, consider the bare algebra $I^0 \subset I$, where $I^0 = \ker(\epsilon: I \rightarrow \mathbb{C})$, which is closed under the action ρ of H , since I is a coalgebra module. Also recall the pre-crossed module of bare algebras $\widehat{\mathcal{BA}}(\mathcal{H}) = (t: I^0 \otimes_{\rho} H \rightarrow H, \triangleright, \triangleleft)$. We will also consider its associated bare algebra crossed module $\mathcal{BA}(\mathcal{H}) = \mathcal{R}(\widehat{\mathcal{BA}}(\mathcal{H})) = (t: \underline{I^0 \otimes_{\rho} H} \rightarrow H, \triangleright, \triangleleft)$; see 2.2.2. Recall that we have algebra maps:

$$\begin{aligned}
i_I: u \in I &\mapsto u \otimes 1_H \in I \otimes_{\rho} H \\
i_H: x \in H &\mapsto 1_I \otimes x \in I \otimes_{\rho} H.
\end{aligned}$$

Clearly, i_I restricts to a map $i_H: \text{Prim}(I) \rightarrow I^0 \otimes_{\rho} H$.

One has a sesquigroupoid $\hat{\mathcal{C}}^+(\widehat{\mathcal{BA}}(\mathcal{H}))$ and a 2-groupoid $\mathcal{C}^+(\mathcal{BA}(\mathcal{H}))$; 2.3.2. Recall Definition 56. The blur holonomy of an \mathcal{H} -valued fully-primitive Hopf 2-connection (ω, m_1, m_2) takes values in $\hat{\mathcal{C}}^+(\widehat{\mathcal{BA}}(\mathcal{H}))$, whereas its fuzzy holonomy takes values in $\mathcal{C}^+(\mathcal{BA}(\mathcal{H}))$. Our starting point is equations (40) and (27).

Definition 71 (Blur Holonomy and Fuzzy Holonomy). Let M be a manifold, provided with a Hopf 2-connection (ω, m_1, m_2) , whose graded form is $(\overline{\omega}, \overline{m})$, taken to be fully primitive (it suffices that both m_1 and m_2 take values in $I^0 \subset I$, however being essential that ω be primitive.) We can see $(i_H(\omega), i_I(m_1), i_I(m_2))$, whose graded form we denote by $i(\overline{\omega}, \overline{m})$, as being a bare 2-connection with values in the pre-crossed module $\widehat{\mathcal{BA}}(\mathcal{H})$ or, projecting onto $\underline{I^0 \otimes_{\rho} H}$, as a bare 2-connection with values in the quotient crossed module $\mathcal{BA}(\mathcal{H})$. The two-dimensional holonomy (as a bare 2-connection) of the former will be called the blur holonomy $\hat{\mathcal{Z}}_{(\overline{\omega}, \overline{m})}^+: \mathcal{P}(M) \rightarrow \hat{\mathcal{C}}^+(\widehat{\mathcal{BA}}(\mathcal{H}))$, of $(\overline{\omega}, \overline{m})$. We call the two-dimensional holonomy of the latter the fuzzy holonomy $\mathcal{Z}_{(\overline{\omega}, \overline{m})}^+: \mathcal{P}(M) \rightarrow \mathcal{C}^+(\mathcal{BA}(\mathcal{H}))$, of (ω, m_1, m_2) .

Therefore, looking at 4.3.1, the blur holonomy of a 2-path $x \begin{array}{c} \xrightarrow{\gamma_1} \\ \uparrow \Gamma \\ \xrightarrow{\gamma_0} \end{array} y$ has the form below:

$$\hat{\mathcal{Z}}_{(\bar{\omega}, \bar{m})}^+ \left(x \begin{array}{c} \xrightarrow{\gamma_1} \\ \uparrow \Gamma \\ \xrightarrow{\gamma_0} \end{array} y \right) = * \begin{array}{c} \xrightarrow{(P_{\bar{\omega}}(\gamma_2))^{-1}} \\ \left[\boxed{+} \right] \uparrow R_{(\bar{\omega}, \bar{m})}(\Gamma) \\ \xrightarrow{(P_{\bar{\omega}}(\gamma_1))^{-1}} \end{array} * \quad (54)$$

where, using the notation of Theorem 17, and given that one-dimensional holonomies are group-like:

$$\begin{aligned} R_{(\bar{\omega}, \bar{m})}(\Gamma) &= - \int_0^1 \int_0^1 i_H(P_{\bar{\omega}}(\Gamma, [0, u], s)) i_I \left(\bar{m} \left(\frac{\partial}{\partial s} \Gamma(u, s), \frac{\partial}{\partial u} \Gamma(u, s) \right) \right) i_H(P_{\bar{\omega}}(\Gamma, [u, 1], s)) du ds \\ &= - \int_0^1 \int_0^1 \left(P_{\bar{\omega}}(\Gamma, [0, u], s) \triangleright \bar{m} \left(\frac{\partial}{\partial s} \Gamma(u, s), \frac{\partial}{\partial u} \Gamma(u, s) \right) \right) \otimes P_{\bar{\omega}}(\Gamma, [0, 1], s) du ds \in (I^0 \otimes_{\rho} H)[[h]]. \end{aligned} \quad (55)$$

Recall that if $\hat{A} = (\partial: A \rightarrow B, \triangleright, \triangleleft)$ is a pre-crossed module of bare algebras, then the associated crossed module of Hopf algebras is denoted by $\mathcal{R}(A) = (\partial: \underline{A} \rightarrow B, \triangleright, \triangleleft)$; see 2.2.2. The algebra projection $A \rightarrow \underline{A}$ is denoted by $a \mapsto \underline{a}$. The fuzzy holonomy $\mathcal{Z}_{(\bar{\omega}, \bar{m})}^+$, of the Hopf 2-connection (ω, m_1, m_2) , which is a projection of the seemingly much finer, blur holonomy $\hat{\mathcal{Z}}_{(\bar{\omega}, \bar{m})}^+$, has the form:

$$\mathcal{Z}_{(\bar{\omega}, \bar{m})}^+ \left(x \begin{array}{c} \xrightarrow{\gamma_1} \\ \uparrow \Gamma \\ \xrightarrow{\gamma_0} \end{array} y \right) = * \begin{array}{c} \xrightarrow{(P_{\bar{\omega}}(\gamma_2))^{-1}} \\ \left[\boxed{+} \right] \uparrow R_{(\bar{\omega}, \bar{m})}(\Gamma) \\ \xrightarrow{(P_{\bar{\omega}}(\gamma_1))^{-1}} \end{array} * \quad (56)$$

We have put:

$$\underline{R}_{(\bar{\omega}, \bar{m})}(\Gamma) = - \int_0^1 \int_0^1 \left(P_{\bar{\omega}}(\Gamma, [0, u], s) \triangleright \bar{m} \left(\frac{\partial}{\partial s} \Gamma(u, s), \frac{\partial}{\partial u} \Gamma(u, s) \right) \right) \otimes P_{\bar{\omega}}(\Gamma, [0, 1], s) du ds. \quad (57)$$

By construction, the fuzzy holonomy $\mathcal{Z}_{(\bar{\omega}, \bar{m})}^+ : \mathcal{P}(M) \rightarrow \mathcal{C}^+(\mathcal{BA}(\mathcal{H}))$ of $(\bar{\omega}, \bar{m})$ preserves all compositions in $\mathcal{P}(M)$ and in $\mathcal{C}^+(\mathcal{BA}(\mathcal{H}))$. The blur holonomy may not preserve the horizontal compositions of 2-morphisms.

4.4.1 The relation between the exact and fuzzy holonomies

There is no evident general relation between the blur $\hat{\mathcal{Z}}_{(\bar{\omega}, \bar{m})}$ and exact $\mathcal{F}_{(\bar{\omega}, \bar{m})}$ holonomies of a fully primitive Hopf 2-connection (ω, m_1, m_2) . Nevertheless, the fuzzy holonomy can be derived from the exact holonomy.

Let $\mathcal{H} = (\partial: I \rightarrow H, \rho)$ be a crossed module of Hopf algebras. Consider the crossed module inclusion of 2.5.7, a map $\text{Inc}_{\mathcal{H}}: (\mathcal{H})_{\text{gl}}^* \rightarrow \mathcal{BA}(\mathcal{H})_{\bullet}^*$ of crossed modules of groups, therefore inducing a map of 2-groupoids:

$$\mathcal{C}^{\times}((\mathcal{H})_{\text{gl}}^*) \xrightarrow{\mathcal{C}^{\times}(\text{Inc}_{\mathcal{H}})} \mathcal{C}^{\times}(\mathcal{BA}(\mathcal{H})_{\bullet}^*).$$

The following theorem is one of the main results of this paper. Recall the notation of Theorem 15:

Theorem 72. *Let (ω, m_1, m_2) be a fully primitive Hopf 2-connection. The diagram below commutes:*

$$\begin{array}{ccc} \mathcal{C}^{\times}((\mathcal{H})_{\text{gl}}^*) & \xleftarrow{\mathcal{F}_{(\bar{\omega}, \bar{m})}^{\times}} & \mathcal{P}(M) \\ \downarrow \mathcal{C}^{\times}(\text{Inc}_{\mathcal{H}}) & & \downarrow \mathcal{Z}_{(\bar{\omega}, \bar{m})}^+ \\ \mathcal{C}^{\times}(\mathcal{BA}(\mathcal{H})_{\bullet}^*) & \xrightarrow{\mathcal{T}_{\mathcal{BA}(\mathcal{H})}} & \mathcal{C}^+(\mathcal{BA}(\mathcal{H})) \end{array} \quad (58)$$

Proof: At the level of 1-paths this is tautological. Recall the construction of 4.2.1. Let $\Gamma = x \begin{array}{c} \xrightarrow{\gamma_1} \\ \uparrow \Gamma \\ \xrightarrow{\gamma_0} \end{array} y$ be a 2-path, and as usual put $\gamma_s(t) = \Gamma(t, s)$. For each $s \in [0, 1]$, consider the 2-morphisms below, of $\mathcal{C}^+(\mathcal{BA}(\mathcal{H}))$:

$$* \begin{array}{c} \xrightarrow{P_{\overline{\omega}}(\gamma_s)^{-1}} \\ \boxed{+} \uparrow \left((Q_{(\overline{\omega}, \overline{m})}(\Gamma, [0, s]) - 1) \otimes P_{\overline{\omega}}(\gamma_0)^{-1} \right) \\ \xrightarrow{P_{\overline{\omega}}(\gamma_0)^{-1}} \end{array} * \quad \text{and} \quad * \begin{array}{c} \xrightarrow{P_{\overline{\omega}}(\gamma_s)^{-1}} \\ \boxed{+} \uparrow R_{(\overline{\omega}, \overline{m})}(\Gamma, [0, s]) \\ \xrightarrow{P_{\overline{\omega}}(\gamma_0)^{-1}} \end{array} * .$$

These coincide with $\mathcal{T}_{\mathcal{BA}(\mathcal{H})} \circ \mathcal{C}^\times(\text{Inc}\mathcal{H}) \circ \mathcal{F}_{(\overline{\omega}, \overline{m})}^\times(\Gamma)$ and $\mathcal{Z}_{(\overline{\omega}, \overline{m})}^+(\Gamma)$, respectively, for $s = 1$; see equations (29) and (46). We prove that, for each $s \in [0, 1]$ (and thus for $s = 1$), we have:

$$\underline{(Q_{(\overline{\omega}, \overline{m})}(\Gamma, [0, s]) - 1) \otimes (P_{\overline{\omega}}(\gamma_0)^{-1})} = \underline{R_{(\overline{\omega}, \overline{m})}(\Gamma, [0, s])}.$$

By definition, and using the second Peiffer law for crossed module of bare algebras in the fourth equality:

$$\begin{aligned} \underline{\frac{d}{ds} (Q_{(\overline{\omega}, \overline{m})}(\Gamma, [0, s]) - 1) \otimes P_{\overline{\omega}}(\gamma_0)^{-1}} &= \underline{\left(\frac{d}{ds} Q_{(\overline{\omega}, \overline{m})}(\Gamma, [0, s]) \right) \otimes P_{\overline{\omega}}(\gamma_0)^{-1}} \\ &= - \underline{\left(\left(\int_0^1 P_{\overline{\omega}}(\Gamma, [0, u], s) \triangleright \overline{m} \left(\frac{\partial}{\partial s} \Gamma(u, s), \frac{\partial}{\partial u} \Gamma(u, s) \right) du \right) Q_{(\overline{\omega}, \overline{m})}(\Gamma, [s, s_0]) \right) \otimes P_{\overline{\omega}}(\gamma_0)^{-1}} \\ &= - \underline{\left(\int_0^1 P_{\overline{\omega}}(\Gamma, [0, u], s) \triangleright \overline{m} \left(\frac{\partial}{\partial s} \Gamma(u, s), \frac{\partial}{\partial u} \Gamma(u, s) \right) du \right) (Q_{(\overline{\omega}, \overline{m})}(\Gamma, [s, s_0]) \otimes P_{\overline{\omega}}(\gamma_0)^{-1}} \\ &= - \underline{\left(\int_0^1 P_{\overline{\omega}}(\Gamma, [0, u], s) \triangleright \overline{m} \left(\frac{\partial}{\partial s} \Gamma(u, s), \frac{\partial}{\partial u} \Gamma(u, s) \right) du \right) \otimes \partial((Q_{(\overline{\omega}, \overline{m})}(\Gamma, [s, s_0]))) (P_{\overline{\omega}}(\gamma_0)^{-1})} \\ &= - \underline{\left(\int_0^1 P_{\overline{\omega}}(\Gamma, [0, u], s) \triangleright \overline{m} \left(\frac{\partial}{\partial s} \Gamma(u, s), \frac{\partial}{\partial u} \Gamma(u, s) \right) du \right) \otimes \partial((Q_{(\overline{\omega}, \overline{m})}(\Gamma, [s, s_0]) P_{\overline{\omega}}(\gamma_0)^{-1})} \\ &= - \underline{\left(\int_0^1 P_{\overline{\omega}}(\Gamma, [0, u], s) \triangleright \overline{m} \left(\frac{\partial}{\partial s} \Gamma(u, s), \frac{\partial}{\partial u} \Gamma(u, s) \right) du \right) \otimes P_{\overline{\omega}}(\gamma_s)^{-1},} \end{aligned}$$

since \overline{m} is primitive and using (47). Whereas, by (57), noting that $P_{\overline{\omega}}(\Gamma, [0, 1], s) = P_{\overline{\omega}}(\gamma_s)$:

$$\begin{aligned} \underline{\frac{d}{ds} (R_{(\overline{\omega}, \overline{m})}(\Gamma, [0, s]))} &= - \underline{\left(\int_0^1 \left(P_{\overline{\omega}}(\Gamma, [0, u], s) \triangleright \overline{m} \left(\frac{\partial}{\partial s} \Gamma(u, s), \frac{\partial}{\partial u} \Gamma(u, s) \right) du \right) \otimes P_{\overline{\omega}}(\Gamma, [0, 1], s) \right)} \\ &\doteq \underline{\left(\int_0^1 P_{\overline{\omega}}(\Gamma, [0, u], v) \triangleright \overline{m} \left(\frac{\partial}{\partial s} \Gamma(u, s), \frac{\partial}{\partial u} \Gamma(u, s) \right) du \right) \otimes P_{\overline{\omega}}(\gamma_s)^{-1}.} \end{aligned}$$

Thence $\underline{(Q_{(\overline{\omega}, \overline{m})}(\Gamma, [0, s]) - 1) \otimes P_{\overline{\omega}}(\gamma_0)^{-1}} = \underline{R_{(\overline{\omega}, \overline{m})}(\Gamma, [0, s])}$, for all $s \in [0, 1]$. \square

4.4.2 The relation between the exact and bare holonomies

Now consider a unital crossed module $\mathcal{A} = (\partial: A \rightarrow B, \triangleright, \triangleleft)$, of bare algebras. From \mathcal{A} we can define the group crossed module $\mathcal{A}_\bullet^* = (\partial: A_\bullet^* \rightarrow B^*, \triangleright)$, as explained in 2.2.4. We can also form (see 2.2.3) the differential crossed module $\text{Lie}(\mathcal{A}) = (\partial: \text{Lie}(A) \rightarrow \text{Lie}(B), \triangleright)$, and, therefore (see 2.5.4) the crossed module of Hopf algebras $\mathcal{U}(\text{Lie}(\mathcal{A})) = (\partial: \mathcal{U}(\text{Lie}(A)) \rightarrow \mathcal{U}(\text{Lie}(B)), \rho)$. From $\mathcal{U}(\text{Lie}(\mathcal{A}))$, we define (see 2.4.4) the crossed module of groups $\mathcal{U}(\text{Lie}(\mathcal{A}))_{\text{gl}}^* = (\partial: \mathcal{U}(\text{Lie}(A))_{\text{gl}}^* \rightarrow \mathcal{U}(\text{Lie}(B))_{\text{gl}}^*, \triangleright)$. By Theorem 34, we have a crossed module map $\text{Proj}_{\mathcal{A}} = (\xi_A^b, \xi_B): \mathcal{U}(\text{Lie}(\mathcal{A}))_{\text{gl}}^* \rightarrow \mathcal{A}_\bullet^*$, called the crossed module counit. We have (see 2.3.1) 2-groupoids $\mathcal{C}^\times(\mathcal{A}_\bullet^*)$ and $\mathcal{C}^\times(\mathcal{U}(\text{Lie}(\mathcal{A}))_{\text{gl}}^*)$. The crossed module counit $\text{Proj}_{\mathcal{A}}: \mathcal{U}(\text{Lie}(\mathcal{A}))_{\text{gl}}^* \rightarrow \mathcal{A}_\bullet^*$ induces a 2-functor $\mathcal{C}^\times(\mathcal{U}(\text{Lie}(\mathcal{A}))_{\text{gl}}^*) \xrightarrow{\mathcal{C}^\times(\text{Proj}_{\mathcal{A}})} \mathcal{C}^\times(\mathcal{A}_\bullet^*)$.

Recall 2.5.6 that from the Hopf algebra crossed module $\mathcal{U}(\text{Lie}(\mathcal{A}))$ we can define a crossed module of bare algebras $\mathcal{BA}(\mathcal{U}(\text{Lie}(\mathcal{A}))) = (t: \mathcal{U}^0(\text{Lie}(\mathcal{A})) \otimes_\rho \mathcal{U}(\text{Lie}(B)) \rightarrow \mathcal{U}(\text{Lie}(B)), \triangleright, \triangleleft)$. By using the construction in 2.3.2, we have 2-groupoids $\mathcal{C}^+(\mathcal{BA}(\mathcal{U}(\text{Lie}(\mathcal{A}))))$ and $\mathcal{C}^+(\mathcal{A})$. Theorem 36 gives us a map of crossed

modules of bare algebras: $K_A \doteq (\kappa_A, \xi_B): \mathcal{BA}(\mathcal{U}(\text{Lie}(\mathcal{A}))) \rightarrow \mathcal{A}$, the bare counit, thus a 2-groupoid map $\mathcal{C}^+(\mathcal{BA}(\mathcal{U}(\text{Lie}(\mathcal{A})))) \xrightarrow{\mathcal{C}^+(K_A)} \mathcal{C}^+(\mathcal{A})$.

If $(\bar{\omega}, \bar{m})$ is a bare 2-connection in \mathcal{A} , seen as a pair of graded forms, we can also look at it as being a fully primitive Hopf 2-connection in $\mathcal{U}(\text{Lie}(\mathcal{A}))$. We will thus consider the exact and fuzzy holonomies $\mathcal{F}_{(\bar{\omega}, \bar{m})}^\times$ and $\mathcal{Z}_{(\bar{\omega}, \bar{m})}^+$, of $(\bar{\omega}, \bar{m})$. These define maps $\mathcal{F}_{(\bar{\omega}, \bar{m})}^\times: \mathcal{P}(M) \rightarrow \mathcal{C}^\times(\mathcal{U}(\text{Lie}(\mathcal{A}))_{\text{gl}}^*)$ and $\mathcal{Z}_{(\bar{\omega}, \bar{m})}^+: \mathcal{P}(M) \rightarrow \mathcal{C}(\mathcal{BA}(\mathcal{U}(\text{Lie}(\mathcal{A}))))$, preserving all compositions, boundaries and inverses. Of course, we can also consider the bare holonomy $\mathcal{F}_{(\bar{\omega}, \bar{m})}^+: \mathcal{P}(M) \rightarrow \mathcal{C}^+(\mathcal{A})$ of $(\bar{\omega}, \bar{m})$.

Now for one of the main theorems of this paper. With a different formulation, and in the case of crossed modules derived from chain-complexes of vector spaces, this was mentioned in [CFM12a] and in [AS14a] (the latter reference giving a detailed proof).

Theorem 73. *The following diagram commutes (once again recall Theorem 15):*

$$\begin{array}{ccc}
 & \mathcal{P}(M) & \\
 \mathcal{F}_{(\bar{\omega}, \bar{m})}^\times \swarrow & & \searrow \mathcal{Z}_{(\bar{\omega}, \bar{m})}^+ \\
 \mathcal{C}^\times(\mathcal{U}(\text{Lie}(\mathcal{A}))_{\text{gl}}^*) & & \mathcal{C}^+(\mathcal{BA}(\mathcal{U}(\text{Lie}(\mathcal{A})))) \\
 \mathcal{C}^\times(\text{Proj}_{\mathcal{A}}) \downarrow & \mathcal{F}_{(\bar{\omega}, \bar{m})}^+ \curvearrowright & \downarrow \mathcal{C}^+(K_A) \\
 \mathcal{C}^\times(\mathcal{A}_\bullet^*) & \xrightarrow{\tau_{\mathcal{A}}} & \mathcal{C}^+(\mathcal{A})
 \end{array}$$

Proof: This follows directly from diagram (17), in the Introduction. The most difficult part of it (the commutativity of the square (58)) was already proven. All of the remaining polygons of the diagram (17), commute either by naturality, definition or by using Lemma 53. \square

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