eprints@whiterose.ac.uk
https://eprints.whiterose.ac.uk/

An Expanding and Shift Scheme for Constructing Fourth-Order Difference Co-Arrays

Jingjing Cai, Wei Liu, Ru Zong, and Qing Shen

Abstract—An expanding and shift (EAS) scheme for efficient fourth-order difference co-array construction is proposed. It consists of two sparse sub-arrays, where one of them is modified and shifted according to the analysis provided. The number of consecutive lags of the proposed structure at the fourth order is consistently larger than two previously proposed methods. Two effective construction examples are provided with the second sparse sub-array chosen to be a two-level nested array, as such a choice can increase the number of consecutive lags further. Simulations are performed to show the improved performance by the proposed method in comparison with existing structures.

Index Terms—Sparse arrays, fourth-order difference co-array, second-order difference co-array, cumulant.

I. INTRODUCTION

Recently, the sparse array concept combined with co-array equivalence has attracted significant interest in the community [1], [2], and two representative examples are the co-prime arrays [3]–[5], and the nested arrays [6], [7]. Sparse arrays can form a larger aperture size given the same number of antennas and more importantly provide much more degrees of freedom (DOFs) than traditional uniform arrays. Many methods have been proposed for underdetermined DOA estimation based on such arrays, such as the spatial smoothing-based subspace methods [8], [9], or compressive sensing (CS)-based methods [4], [10]–[13].

So far the majority of work for virtual array generation is based on the second-order statistics (SOS). However, it is possible to exploit the fourth-order statistics (FOS) to generate even more DOFs, such as the cumulant-based DOA estimation methods studied in [14]–[20] and the method based on quasi-stationary signals [21], [22]. Therefore, how to construct a sparse array with maximum fourth-order virtual array sensors has become a very important problem. In [22], [23], the existing nested arrays and co-prime arrays were extended for effective fourth-order virtual array generation by adding to the structure a third uniform linear array. It was shown that in

Jingjing Cai and Ru Zong are with the Department of Electronic and Engineering, Xidian University, Xi'an, 710071, China (e-mail: {jjcai,zongru}@mail.xidian.edu.cn). Jingjing Cai is also a visiting researcher at the Department of Electronic and Electrical Engineering, University of Sheffield, Sheffield, S1 3JD, UK.
Wei Liu is with the Department of Electronic and Electrical Engineering, University of Sheffield, Sheffield, S1 3JD, UK (e-mail: w.liu@sheffield.ac.uk).
Qing Shen is with the School of Information and Electronics, Beijing Institute of Technology, Beijing, 100081, China (e-mail: qing-shen@outlook.com).

this way, the number of consecutive virtual sensor lags can be increased significantly.

In this work, by analysing the generated fourth-order difference lags, we can consider them as the sum of two second-order difference lags with the same range. Based on this observation, we propose a new sparse array construction scheme aiming to maximize the consecutive lags in the fourth order virtual co-array.

We start from two separate sparse sub-arrays, and each of them is configured at the second-order difference co-array (SODCA) level, such as the existing co-prime arrays or nested arrays. Then one of them is expanded uniformly by increasing the adjacent physical sensor spacing according to the number of second-order consecutive virtual array sensors of the other sparse sub-array. The last step is shifting the newly expanded array to a new position so that the number of fourth-order consecutive virtual array sensors is further increased and the first sensor of the expanded sub-array coincides with one of the physical sensors of the other sub-array. Due to the coincidence, one of the two physical sensors can be removed without affecting the resultant fourth-order DOFs in the consecutive range. It is also shown that if the second sub-array is a two-level nested array, the fourth-order consecutive virtual sensor range can be further increased. Compared with the fourth-order difference co-arrays (FODCAs) proposed in [22], [23], a higher number of consecutive lags is achieved by the proposed scheme.

This paper is organized as follows. The cumulant-based FODCAs are analyzed in Sec. II, and the proposed construction is introduced in detail in Sec. III. A comparison of the different fourth-order construction schemes is performed in Sec. IV. Simulation results are provided in Sec. V and conclusion drawn in Sec. VI.

II. CUMULANT-BASED FOURTH-ORDER DIFFERENCE CO-ARRAY

Suppose there are K far-field independent non-Gaussian narrowband signals $s_k(t)(k = 1, \ldots, K)$ impinging on a sparse linear array (SLA) of M sensors. Denote the unit spacing by d, which is equal to half wavelength $\lambda/2$. Then, positions of the SLA sensors can be expressed as

$$P = \{p_1 \cdot d, p_2 \cdot d, \ldots, p_m \cdot d, \ldots, p_M \cdot d\} . \quad (1)$$

With the angle of arrival of the kth source being θ_k, the observed signal $x_m(t)$ at the mth sensor is given by

$$x_m(t) = \sum_{k=1}^{K} exp(-j2\pi p_m d cos\theta_k/\lambda)s_k(t) + n_m(t) , \quad (2)$$

where $n_m(t)$ is the additive Gaussian noise of the mth sensor, and it is independent of the signals. Suppose $1 \leq i, j, u, v \leq M$ and $\{i, j, u, v\} \in Z$. The fourth-order cumulant value $C(i, -j, u, -v)$ of the ith, jth, uth and vth sensor observed signals can be expressed as [14]

$$
\begin{aligned}
C(i, -j, u, -v) &= cum[x_i(t), x_j^*(t), x_u(t), x_v^*(t)] \\
&= \sum_{k=1}^{K} exp[-j2\pi(p_i - p_j + p_u - p_v)dcos\theta_k/\lambda] \cdot \\
&\quad cum(x_1(t), x_1^*(t), x_1(t), x_1^*(t))
\end{aligned}
\tag{3}
$$

where $()^*$ denotes complex conjugate, and $cum()$ denotes the fourth-order cumulant operation. The fourth-order difference co-array not only has a much larger number of virtual sensors than the physical SLA, but also removes the Gaussian noise components, which will help improving the DOA estimation result further.

The fourth-order difference lag $(p_i - p_j + p_u - p_v)$ corresponding to the new virtual sensors can be written as

$$
p_i - p_j + p_u - p_v = (p_i - p_j) + (p_u - p_v)
\tag{4}
$$

Clearly, this fourth-order difference lag expression can be seen as two second-order difference lags added together. As a result, we could construct the FODCA by two separate SODCAs with different ranges. Although this will not be optimal, it could provide an effective solution for FODCA construction, as shown later.

III. PROPOSED EXPANDING AND SHIFT (EAS) SCHEME FOR CONSTRUCTING FODCAS

A. The EAS Scheme

We start from two separate sparse sub-arrays which are configured for SODCA generation. They can be different types of sparse arrays and have different number of physical sensors. Assume the first sub-array contains M physical sensors, while the second one contains N sensors. Their array position settings are defined as

$$
\begin{aligned}
P &= \{p_1 \cdot d, p_2 \cdot d, \ldots, p_m \cdot d, \ldots, p_M \cdot d\} \\
Q &= \{q_1 \cdot d, q_2 \cdot d, \ldots, q_n \cdot d, \ldots, q_N \cdot d\}
\end{aligned}
\tag{5}
$$

where $p_m \cdot d$ and $p_n \cdot d$, $m = 1, 2, \ldots, M$, $n = 1, 2, \ldots, N$, are the physical sensor positions of the two sub-arrays. We further assume that the number of consecutive SODCA sensors for the first sub-array P is C_M, while it is C_N for Q.

Based on the two sparse arrays, we can generate the fourth-order difference lags using the expression $(p_i - p_j) + (p_u - p_v)$ in (4), where one choice is that the lags $(p_i - p_j)$ come from the second-order co-array of P, while $(p_u - p_v)$ from that of Q. When adding these two together, the segment of C_M consecutive virtual sensors from P are then shifted one by one by the second-order virtual co-array sensors of Q, and there are at least C_N copies of the same C_M consecutive points from P, which are then added together to form the FODCA. To make sure there are no overlaps or gaps among the C_N copies of the continuous segment of length C_M so that the maximum number of consecutive fourth-order virtual sensors are achieved, we can increase the unit spacing of the sub-array

Q to $C_M \cdot d$. As a result, the second sparse array is changed to

$$
\tilde{Q} = \{q_1, q_2, \ldots, q_n, \ldots, q_N\} \cdot C_M d ,
\tag{6}
$$

and the number of consecutive fourth-order virtual co-array sensors is $C_L = C_M C_N$.

Note that the C_L consecutive lags in the fourth-order co-array is independent of the relative positions of the two sparse arrays P and $\tilde{Q}$, since any shift will be canceled by the operation of $(p_i - p_j)$ and $(p_u - p_v)$. As a result, we can shift the starting position of the second array $\tilde{Q}$ by $\Delta s \cdot d$ so that one of the physical sensors of the second sparse array will be co-located with one of the physical sensors of the first array, i.e. $q_n C_M = p_m$ for a specific pair of (m, n). Then, one of the co-located sensors can be removed and the total number of physical sensors will be $L = M + N - 1$ with the same number of consecutive fourth-order co-array sensors C_L. To have a larger aperture, we can choose $q_1 C_M = p_m$, i.e. the first sensor of the second array will coincide with one arbitrary sensor of the first array. Without loss of generality, we remove the first sensor of the second array $\tilde{Q}$. Then, the pair of sparse sub-arrays becomes

$$
\begin{aligned}
P &= \{p_1 \cdot d, p_2 \cdot d, \ldots, p_m \cdot d, \ldots, p_M \cdot d\} \\
\hat{Q} &= \{q_2, \ldots, q_n, \ldots, q_N\} \cdot C_M d ,
\end{aligned}
\tag{7}
$$

Interestingly, as we will show in the next part, this choice of shift will have the advantage of generating additional consecutive lags of $2(p_m - p_1)$ if the second sparse array is chosen to be a two-level nested array (referred to as EAS-NA in the following). Under this condition, if the first array is further chosen to be either a nested array or a co-prime array, we can have $q_1 C_M = p_M$, so that the total number of consecutive fourth-order co-array sensors becomes $C_L = C_M C_N + 2(p_M - p_1)$ for the EAS-NA scheme. The physical array aperture of the proposed construction scheme is $(q_N C_M - p_1)d$ for the general case of $q_1 C_M = p_m$.

B. The EAS-NA Scheme

In this part, we consider the EAS-NA scheme, as it will increase the fourth-order consecutive lags further.

For a nested array, we have the interesting property of $q_N - q_1 = \frac{C_N - 1}{2}$, where $\frac{C_N - 1}{2}$ is the maximum number of positive consecutive second-order lags. For such an EAS-NA construction, the range of the positive consecutive fourth-order lags is from 1 to $(q_N - q_1)C_M + \frac{C_M - 1}{2}$, and the last segment of C_M consecutive fourth-order lags is from $(q_N - q_1)C_M - \frac{C_M - 1}{2}$ to $(q_N - q_1)C_M + \frac{C_M - 1}{2}$, centered at $(q_N - q_1)C_M$. With $q_1 C_M = p_M$ as suggested in the last subsection, this center becomes $q_N C_M - p_M$

Note that the last sensor of $\tilde{Q}$ is $q_N C_M$. When calculating the fourth-order difference lag $(p_i - p_j) + (p_u - p_v)$, p_i and p_j can be chosen from the first array P, while $p_u = q_N C_M$ and $p_v = p_1$. For such a choice, $(p_i - p_j)$ will general a segment of consecutive lags from $-\frac{C_M - 1}{2}$ to $\frac{C_M - 1}{2}$, so that $(p_i - p_j) + (p_u - p_v)$ in total will generate a segment of consecutive lags from $q_N C_M - p_1 - \frac{C_M - 1}{2}$ to $q_N C_M - p_1 + \frac{C_M - 1}{2}$, centered at $q_N C_M - p_1$.

Now we have two segments of consecutive lags of length C_M, one centered at $q_N C_M - p_M$ and one at $q_N C_M - p_1$, where the second one is $p_M - p_1$ away from the first one. If $p_M - p_1 \leq (C_M - 1)$, then the consecutive fourth-order lags will be increased from $C_M C_N$ to $C_M C_N + 2(p_M - p_1)$, where the multiplication by 2 is due to considering both the negative and the positive consecutive lags. Since the condition $p_M - p_1 \leq (C_M - 1)$ is satisfied for most sparse arrays designed for maximizing the continuous second-order difference co-array lags, such as the co-prime arrays and nested arrays, we can shift the second sparse array so that its first sensor will be co-located with the last sensor of the first array.

C. Examples of the EAS-NA Scheme

Here, we give two examples for the EAS-NA scheme. One takes two nested arrays as its two sparse sub-arrays, referred to as the EAS-NA-NA array; the other one use the co-prime array as its first sparse sub-array and the nested array as the second sparse sub-array, referred to as the EAS-NA-CPA array.

First we consider the EAS-NA-NA case. The first nested array contains M_1 and M_2 sensors in its two sub-arrays, separately, and the second nested array contains N_1 and N_2 sensors. The total number of sensors is $L = M_1 + M_2 + N_1 + N_2 - 1$. The consecutive second-order co-array sensor number of these two nested arrays is $C_M = 2M_1 M_2 + 2M_2 - 1$ and $C_N = 2N_1 N_2 + 2N_2 - 1$, respectively. The first sensor of the first nested array is 1 and the last sensor is $p_M = M_1 M_2 + M_2$. Then, the total number of consecutive fourth-order difference co-array lags is $C_L = (2M_1 M_2 + 2M_2 - 1)(2N_1 N_2 + 2N_2 - 1) + 2(M_1 M_2 + M_2 - 1)$.

As an example, for $M_1 = N_1 = 2$ and $N_1 = N_2 = 2$, we have $C_M = C_N = 11$ and $C_L = 131$. The resultant sensor locations are $P = \{1, 2, 3, 6\} \cdot d$ and $\hat{Q} = \{17, 28, 61\} \cdot d$.

Now for the EAS-NA-CPA case, assume the co-prime array contains M_1 and M_2 sensors as its two sub-arrays, separately, and the nested array contains N_1 and N_2 sensors as before. Then, we have $L = 2M_1 + M_2 + N_1 + N_2 - 2$, $C_M = 2M_1 M_2 + 2M_1 - 1$ and $C_N = 2N_1 N_2 + 2N_2 - 1$. The first sensor of the co-prime array is 0 and the last sensor is $p_M = 2M_1 M_2 - M_2$. As a result, we can obtain $C_L = (2M_1 M_2 + 2M_1 - 1)(2N_1 N_2 + 2N_2 - 1) + 2(2M_1 M_2 - M_2)$.

With $M_1 = 2, M_2 = 3, N_1 = 2$ and $N_2 = 2$. The results are $C_M = 15$, $C_N = 11$ and $C_L = 183$. The set of sensor locations are $P = \{0, 2, 3, 4, 6, 9\} \cdot d$ and $\hat{Q} = \{24, 39, 84\} \cdot d$.

IV. COMPARISON BETWEEN DIFFERENT CO-ARRAY STRUCTURES

In this section, we give a comparison between our proposed schemes (EAS-NA-CPA and EAS-NA-NA as two specific cases) with two recently proposed ones: one is called SAFO-CPA [22], and the other one is called SAFO-NA [23].

Since given the same number of physical sensors, we can have different sub-array parameters, which then results into different number of consecutive fourth-order lags for the same construction scheme. To have a fair comparison, we choose the parameters giving the maximum number for each scheme. Fig. 1 shows the number of consecutive fourth-order lags C_L for
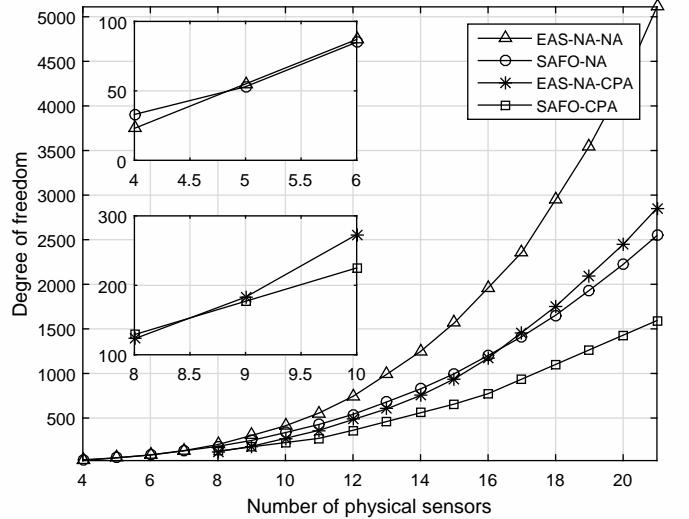

Fig. 1. DOFs of different co-array structures in terms of the number of consecutive FODCA lags.

TABLE I
PARAMETER SETTING OF THE CORRESPONDING SPARSE ARRAYS.

number of sensors	SAFO-NA (M_1, M_2, N)	EAS-NA-NA (M_1, M_2, N_1, N_2)	SAFO-CPA (M_1, M_2, N)	EAS-NA-CPA (M_1, M_2, N_1, N_2)
4	$(1, 2, 1)$	$(1, 1, 1, 2)$	$--$	$--$
5	$(1, 2, 2)$	$(1, 2, 1, 2)$	$--$	$--$
6	$(2, 2, 2)$	$(2, 2, 1, 2)$	$--$	$--$
7	$(2, 3, 2)$	$(2, 3, 1, 2)$	$--$	$--$
8	$(2, 3, 3)$	$(2, 3, 2, 2)$	$(2, 3, 2)$	$(2, 3, 1, 2)$
9	$(3, 3, 3)$	$(2, 3, 2, 3)$	$(2, 3, 3)$	$(2, 3, 2, 2)$
10	$(3, 4, 3)$	$(3, 3, 2, 3)$	$(2, 3, 4)$	$(2, 3, 2, 3)$
11	$(3, 4, 4)$	$(3, 4, 2, 3)$	$(2, 3, 5)$	$(2, 3, 3, 3)$
12	$(4, 4, 4)$	$(3, 4, 3, 3)$	$(3, 4, 3)$	$(2, 3, 3, 4)$
13	$(4, 5, 4)$	$(3, 4, 3, 4)$	$(3, 4, 4)$	$(2, 3, 4, 4)$
14	$(4, 5, 5)$	$(4, 4, 3, 4)$	$(3, 4, 5)$	$(2, 3, 4, 5)$
15	$(5, 5, 5)$	$(4, 5, 3, 4)$	$(3, 4, 6)$	$(3, 4, 3, 4)$
16	$(5, 6, 5)$	$(4, 5, 4, 4)$	$(4, 5, 4)$	$(3, 4, 4, 4)$
17	$(5, 6, 6)$	$(4, 5, 4, 5)$	$(4, 5, 5)$	$(3, 4, 4, 5)$
18	$(6, 6, 6)$	$(5, 5, 4, 5)$	$(4, 5, 6)$	$(3, 4, 5, 5)$
19	$(6, 7, 6)$	$(5, 6, 4, 5)$	$(4, 5, 7)$	$(3, 4, 5, 6)$
20	$(6, 7, 7)$	$(5, 6, 5, 5)$	$(4, 5, 8)$	$(3, 4, 6, 6)$
21	$(7, 7, 7)$	$(5, 6, 5, 6)$	$(5, 6, 6)$	$(3, 4, 6, 7)$

the four cases with different number of physical sensors and the corresponding parameter settings are provided in Tab. IV.

We can see from the figure that, for the total number of physical sensors $L > 4$, the number of DOFs of EAS-NA-NA is always larger than the SAFO-NA structure, while EAS-NA-CPA outperforms the SAFO-CPA for $L > 8$. On the other hand, EAS-NA-CPA and SAFO-NA have a similar result and the C_L number for the EAS-NA-CPA will exceeds that of SAFO-NA for $L > 16$. The performance of EAS-NA-NA is the best of all, which greatly exceeds the other three structures for $L > 10$. For example, for $L = 18$, the number of consecutive fourth-order lags are 2949, 1751, 1653 and 1101 for EAS-NA-NA, EAS-NA-CPA, SAFO-NA, and SAFO-CPA, respectively.

V. SIMULATION RESULTS

In this section, simulations are performed to demonstrate the performance of the proposed EAS-NA scheme. The CS-
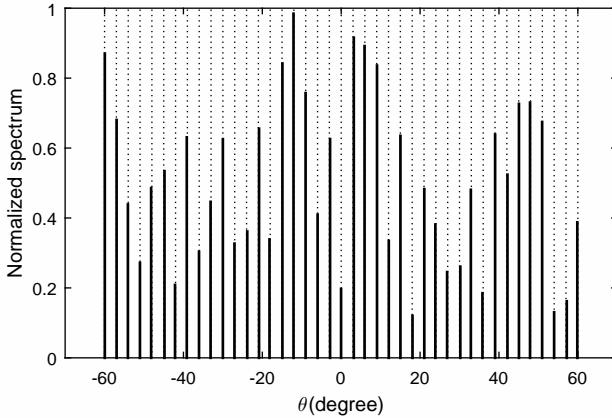
Fig. 2. DOA estimation result for the EAS-NA-NA array.

Fig. 3. RMSE results with respect to input SNR.

Fig. 4. RMSE results with respect to snapshot number.

based DOA estimation algorithm is employed as in [22], [23], where the constrained L_1 norm minimization problem can be solved using cvx, a package for specifying and solving convex problems [24], [25]. In the formulation, the full angle range from $-90°$ to $90°$ is discretized with a step size of $0.05°$. The sources are generated by fixing the magnitude and frequency of a complex baseband signal and then changing its phase randomly following a uniform distribution on $[0, 2\pi]$.

In the first simulation, we consider an EAS-NA-NA array with $L = 6$ sensors and the parameters are set to be $M_1 = 1$ and $M_2 = N_1 = N_2 = 2$, with $P = \{1, 2, 4\} \cdot d$ and $\hat{Q} = \{11, 18, 39\} \cdot d$. $K = 41$ narrowband source signals are uniformly distributed between $-60°$ and $60°$. The input SNR is 0dB, and the number of snapshots for calculating the fourth-order cumulant matrix is 20000. The DOA estimation result is shown in Fig. 2. Clearly, all the sources have been distinguished successfully.

Now we compare the performance of two nested array based structures, EAS-NA-NA and SAFO-NA, and the two co-prime array based structures, EAS-NA-CPA and SAFO-CPA, all with $L = 12$ physical sensors. The parameters for the EAS-NA-NA array are $M_2 = 4$, $M_1 = N_1 = N_2 = 3$, for the SAFO-NA array are $M_1 = M_2 = N = 4$, for the EAS-NA-CPA array are $M_1 = 2$, $M_2 = N_1 = 3$ and $N_2 = 4$, and finally for the SAFO-CPA array are $M_1 = N = 3$, $M_2 = 4$. By calculation, the physical aperture for EAS-NA-NA is $371 \cdot d$, $270 \cdot d$ for SAFO-NA, $241 \cdot d$ for EAS-NA-CPA, and $181 \cdot d$ for SAFO-CPA. The number of source signals is $K = 35$ and the number of snapshots for calculating the fourth-order cumulant matrix is 10000. The root-mean-squared error (RMSE) results obtained through 500 Monte Carlo trials are shown in Fig. 3 with a varied input SNR.

Evidently, the higher the input SNR, the higher its estimation accuracy. The performance of the nested array based structures are better than the co-prime array based ones, while the EAS-NA-NA has achieved the best performance for the whole input SNR range, which is due to not only a higher number of DOFs provided by the EAS-NA structure, but also a larger aperture.

Next we fix the input SNR to 0dB, and change the number of snapshots. The RMSE results are shown in Fig.4, where we can see a similar trend and again the EAS-NA-NA structure has provided the best result for the considered full range of snapshot numbers.
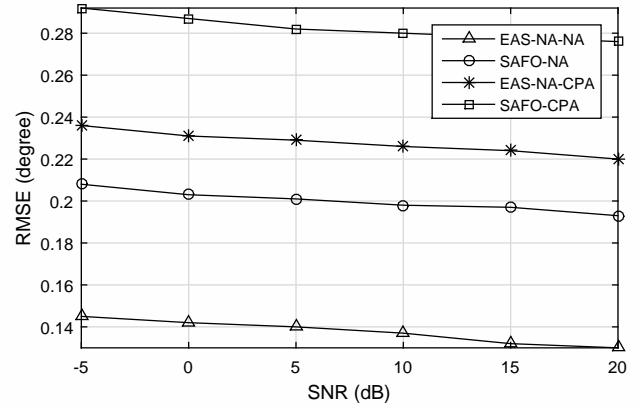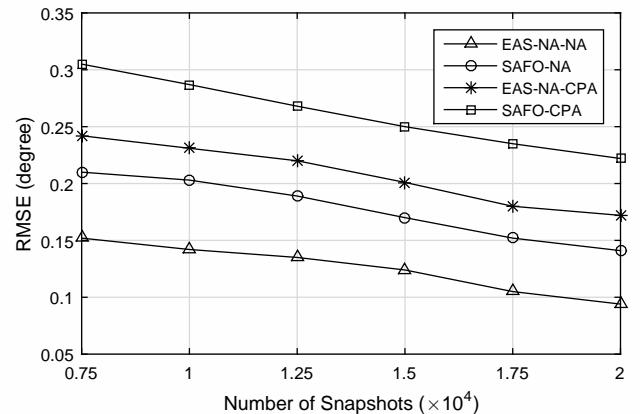
VI. CONCLUSION

A general sparse array construction scheme called expanding and shift (EAS) has been proposed for maximizing the continuous FODCA lags. It consists of two existing sparse sub-arrays, one with M physical sensors and C_M consecutive SODCA lags, while the other one with N physical sensors and C_N consecutive SODCA lags. Then, the second sub-array is first expanded by increasing its unit spacing C_M times and then shifted to a position so that the two sub-arrays share one common physical sensor. As a result, with only $M + N - 1$ physical sensors, $C_M C_N$ consecutive FODCA lags can be achieved. It is also shown that when the second sub-array is a two-level nested array, the number of consecutive FODCA lags can be further increased. As demonstrated by simulation results, the proposed EAS scheme has achieved a much better performance than two existing structures due to its higher number of DOFs and larger physical aperture.

REFERENCES

[1] H. L. Van Trees, *Optimum Array Processing, Part IV of Detection, Estimation, and Modulation Theory*. New York: Wiley, 2002.

[2] R. T. Hoctor and S. A. Kassam, "The unifying role of the coarray in apeture synthesis for coherent and incoherent imaging," *Proceedings of the IEEE*, vol. 78, no. 4, pp. 735 –752, Apr. 1990.

[3] P. Pal and P. P. Vaidyanathan, "Co-prime sampling and the music algorithm," in *IEEE Digital Signal Processing Workshop and IEEE Signal Processing Education Workshop(DSP/SPE)*, Sedona, AZ, January 2011, pp. 289–294.

[4] P. P. Vaidyanathan and P. Pal, "Sparse sensing with co-prime samplers and arrays," *IEEE Transactions on Signal Processing*, vol. 59, no. 2, pp. 573–586, Feb. 2011.

[5] Y. M. Zhang, M. G. Amin, and B. Himed, "Sparsity-based DOA estimation using co-prime arrays," in *Proc. IEEE International Conference on Acoustics, Speech, and Signal Processing*, Vancouver, Canada, May 2013, pp. 3967–3971.

[6] P. Pal and P. P. Vaidyanathan, "Nested arrays: a novel approch to array processing with enhanced degrees of freedom," *IEEE Transactions on Signal Processing*, vol. 58, no. 8, pp. 4167–4181, Aug. 2010.

[7] Z. B. Shen, C. X. Dong, Y. Y. Dong, G. Q. Zhao, and L. Huang, "Broadband DOA estimation based on nested arrays," *International Journal of Antennas and Propagation*, vol. 2015, 2015.

[8] P. Pal and P. P. Vaidyanathan, "Coprime sampling and the MUSIC algorithm," in *Proc. IEEE Digital Signal Processing Workshop and IEEE Signal Processing Education Workshop (DSP/SPE)*, Sedona, AZ, Jan. 2011, pp. 289–294.

[9] C.-L. Liu and P. P. Vaidyanathan, "Remarks on the spatial smoothing step in coarray MUSIC," vol. 22, no. 9, pp. 1438–1442, Sep. 2015.

[10] Q. Shen, W. Liu, W. Cui, S. L. Wu, Y. D. Zhang, and M. Amin, "Low-complexity direction-of-arrival estimation based on wideband co-prime arrays," *IEEE Trans. Audio, Speech and Language Processing*, vol. 23, pp. 1445–1456, September 2015.

[11] S. Qin, Y. D. Zhang, and M. G. Amin, "Generalized coprime array configurations for direction-of-arrival estimation," *IEEE Transactions on Signal Processing*, vol. 63, no. 6, pp. 1377–1390, March 2015.

[12] J. J. Cai, D. Bao, and P. Li, "Doa estimation via sparse recovering from the smoothed covariance vector," *Journal of Systems Engineering and Electronics*, vol. 27, no. 3, pp. 555–561, June 2016.

[13] Q. Shen, W. Liu, W. Cui, and S. L. Wu, "Underdetermined DOA estimation under the compressive sensing framework: A review," *IEEE Access*, vol. 4, pp. 8865–8878, 2016.

[14] J. F. Cardoso and E. Moulines, "Asymptotic performance analysis of direction-finding algorithms based on fourth-order cumulants," *IEEE Transactions on Signal Processing*, vol. 43, no. 1, pp. 214 –224, Jan. 1995.

[15] M. C. Dogan and J. M. Mendel, "Applications of cumulants to array processing.i.aperture extension and array calibration," *IEEE Transactions on Signal Processing*, vol. 43, no. 5, pp. 1200 –1216, May 1995.

[16] P. Chevalier and A. Ferreol, "On the virtual array concept for the fourth-order direction finding promblem," *IEEE Transactions on Signal Processing*, vol. 47, no. 9, pp. 2592 –2595, Sep. 1999.

[17] P. Chevalier, L. Albera, A. Férréol, and P. Comon, "On the virtual array concept for higher order array processing," vol. 53, no. 4, pp. 1254–1271, Apr. 2005.

[18] P. Cardoso and A. Ferreol, "High-resolution direction finding from higher order statistics: The 2q-music algorithm," *IEEE Transactions on Signal Processing*, vol. 54, no. 8, pp. 2986 –2997, Aug. 2006.

[19] G. Birot, L. Albera, and P. Chevalier, "Sequential high-resolution direction finding from higher order statistics," *IEEE Transactions on Signal Processing*, vol. 58, no. 8, pp. 4144–4155, Aug 2010.

[20] P. Pal and P. P. Vaidyanathan, "Multiple level nested array: an efficient geometry for 2qth order cumulant based array processing," *IEEE Transactions on Signal Processing*, vol. 60, no. 3, pp. 1253–1269, Mar. 2012.

[21] W. K. Ma, T. H. Hsieh, and C. Y. Chi, "DOA estimation of quasi-stationary signals with less sensors than sources and unknown spatial noise covariance: a khatri–rao subspace approach," *IEEE Transactions on Signal Processing*, vol. 58, no. 4, pp. 2168–2180, April 2010.

[22] Q. Shen, W. Liu, W. Cui, and S. L. Wu, "Extension of co-prime arrays based on the fourth-order difference co-array concept," *IEEE Signal Processing Letters*, vol. 23, no. 5, pp. 615–619, May 2016.

[23] ——, "Extension of nested arrays with the fourth-order difference co-array enhancement," in *The 41st IEEE International Conference on Acoustics, Speech and Signal Processing*, Shanghai, China, March 2016, pp. 2991–2995.

[24] C. Research, "CVX: Matlab software for disciplined convex programming, version 2.0 beta," http://cvxr.com/cvx, September 2012.

[25] M. Grant and S. Boyd, "Graph implementations for nonsmooth convex programs," in *Recent Advances in Learning and Control*, ser. Lecture Notes in Control and Information Sciences, V. Blondel, S. Boyd, and H. Kimura, Eds. Springer-Verlag Limited, 2008, pp. 95–110, http://stanford.edu/~boyd/graph_dcp.html.