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HONEST ELEMENTARY DEGREES AND DEGREES OF RELATIVE PROVABILITY WITHOUT THE CUPPING PROPERTY

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ABSTRACT. An element a of a lattice *cups* to an element $b > a$ if there is a $c < b$ such that $a \cup c = b$. An element of a lattice has the *cupping property* if it cups to every element above it. We prove that there are non-zero honest elementary degrees that do not have the cupping property, which answers a question of Kristiansen, Schlage-Puchta, and Weiermann [17]. In fact, we show that if $\mathbf{b}$ is a sufficiently large honest elementary degree, then $\mathbf{b}$ has the *anti-cupping property*, which means that there is an $\mathbf{a}$ with $\mathbf{0} <_{\mathrm{E}} \mathbf{a} <_{\mathrm{E}} \mathbf{b}$ that does not cup to $\mathbf{b}$. For comparison, we also modify a result of Cai [8] to show, in several versions of the degrees of relative provability that are closely related to the honest elementary degrees, that in fact all non-zero degrees have the anti-cupping property, not just sufficiently large degrees.

1. Introduction

An element a of a lattice *cups* to an element $b > a$ if there is a $c < b$ such that $a \cup c = b$. An element a of a lattice has the *cupping property* if it cups to every $b > a$. An element b of a lattice with 0 has the *anti-cupping property* if there is an a with $0 < a < b$ that does not cup to b. So, if $a, b > 0$ in a lattice with 0, a witnesses that b has the anti-cupping property if and only if b witnesses that a does not have the cupping property. In this work, we study cupping in several related lattices arising from elementary functions and total algorithms.

The first lattice we consider is the the lattice $\mathcal{H}$ of *honest elementary degrees*, which arose from attempts to classify various sub-recursive classes of functions into hierarchies. In $\mathcal{H}$, the objects are (equivalence classes of) functions whose graphs are elementary relations, and these functions are compared via the 'elementary in' relation. The basic theory of this structure was developed by Meyer and Ritchie [21] and by Machtey [18–20]. In recent years, intense work mainly by Kristiansen [11–17] has significantly advanced the theory. We refer the reader to [16] (and to the related [17]) for a survey of the area. In [17], the authors ask if every non-zero $\mathbf{a} \in \mathcal{H}$ has the cupping property. We answer this question negatively by showing that if $\mathbf{b} \in \mathcal{H}$ is sufficiently large (in the sense of Definition 3.1), then $\mathbf{b}$ has the anti-cupping property (Corollary 3.4). Thus if $\mathbf{b} \in \mathcal{H}$ is sufficiently large, then every $\mathbf{a} \in \mathcal{H}$ witnessing that $\mathbf{b}$ has the anti-cupping property is a non-zero degree that does not have the cupping property.

Next we consider two related families of lattices: the *degrees of provability* relative to arithmetical theories extending $\mathrm{I}\Sigma_1$ and the *honest α-elementary degrees* for ordinals $\alpha \leq \epsilon_0$ of the form ω^β. Let T be a consistent first-order theory in the language of arithmetic. In $\mathcal{P}_T$, the degrees of provability relative to T, the objects are (equivalence classes of) total algorithms (i.e., indices of total Turing machines), and these algorithms are compared via the 'provably total' relation. That is, $\deg_T(\Phi) \geq_T \deg_T(\Psi)$ if $T \vdash \mathrm{tot}(\Phi) \to \mathrm{tot}(\Psi)$, where $\mathrm{tot}(\Phi)$ is the sentence expressing the totality of the Turing machine Φ. Cai [7] introduced the degrees of relative provability in order to analyze the provability strengths of true Π_2 sentences or, equivalently, sentences expressing the totality of total algorithms. This line of research continues impressively in [2, 8, 9].

In $\mathcal{H}_\alpha$, the honest α-elementary degrees, the objects are again (equivalence classes of) functions whose graphs are elementary, and these functions are compared via the 'α-elementary in' relation,

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which coarsens the 'elementary in' relation by allowing functions to be iterated $\beta < \alpha$ many times. Kristiansen, Schlage-Puchta, and Weiermann [17] introduced the honest α-elementary degrees and proved a correspondence between the 'α-elementary in' relation and the 'provably total' relation relative to Peano arithmetic (PA). Such correspondences between sub-recursive hierarchies and provably total functions can be useful for analyzing the logical strengths of formal systems. See, for example, Beklemishev's work in [3–6]

The degrees of relative provability and the honest α-elementary degrees are very closely related. For a theory T, let T^+ be the extension of T by all true Π_1 sentences. Kristiansen [15] proves that $\mathcal{P}_{\mathrm{PA}^+}$ and $\mathcal{H}_{\epsilon_0}$ are isomorphic, and analogous results should hold for various fragments of PA and the appropriate ordinals.

Cai [8] proves that there are non-zero elements of $\mathcal{P}_{\mathrm{PA}^+}$ that do not have the cupping property. It follows from Kristiansen's isomorphism that there are also non-zero elements of $\mathcal{H}_{\epsilon_0}$ that do not have the cupping property. We modify Cai's result to prove that if T is a consistent, recursively axiomatizable theory extending $\mathrm{I}\Sigma_1$, then every non-zero $\mathbf{b} \in \mathcal{P}_{T^+}$ has the anti-cupping property (Corollary 5.3). Consider then the following two statements:

($\star$) Every b that is sufficiently large (where the meaning of 'sufficiently large' depends on the lattice in question) has the anti-cupping property.

($\dagger$) Every $b > 0$ has the anti-cupping property.

Corollary 3.4 states that ($\star$) holds in $\mathcal{H}$. By modifying the argument, we also see that ($\star$) holds in the $\mathcal{H}_\alpha$'s. Corollary 5.3 states that ($\dagger$) holds in $\mathcal{P}_{T^+}$ for every consistent, recursively axiomatizable theory T extending $\mathrm{I}\Sigma_1$. In particular, ($\dagger$) holds in $\mathcal{P}_{\mathrm{PA}^+}$ and so, by Kristiansen's isomorphism, also in $\mathcal{H}_{\epsilon_0}$. Thus the natural question is whether or not ($\dagger$) holds in $\mathcal{H}$ and in every $\mathcal{H}_\alpha$. We expect that ($\dagger$) holds in many of the $\mathcal{H}_\alpha$'s by extending Kristiansen's isomorphism result to fragments of PA.

2. Honest elementary degrees

In this section, we provide a basic introduction to the theory of the honest elementary degrees. Again, we refer the reader to [16, 17] for more comprehensive surveys.

Definition 2.1.

- The *elementary functions* are those functions $f\colon \omega^n \to \omega$ that can be generated from the *initial elementary functions* by the *elementary definition schemes*.
- The *initial elementary functions* are
  - the projection functions ℓ_i^k for all $k > 0$ and $i < k$, where $\ell_i^k(x_0, \dots, x_i, \dots, x_{k-1}) = x_i$;
  - the 0-ary constants 0 and 1; addition ($+$); and truncated subtraction (i.e., monus $\dot{-}$).
- The *elementary definition schemes* are
  - composition: $f(\vec{x}) = h(g_0(\vec{x}), g_1(\vec{x}), \dots, g_{m-1}(\vec{x}))$;
  - bounded sum: $f(\vec{x}, y) = \sum_{i<y} g(\vec{x}, i)$; and
  - bounded product: $f(\vec{x}, y) = \prod_{i<y} g(\vec{x}, i)$.
- A relation is *elementary* if its characteristic function is elementary.
- A function f has *elementary graph* if the relation $R(\vec{x}, y) :\equiv (f(\vec{x}) = y)$ is elementary.
- A function $f\colon \omega^n \to \omega$ is *elementary in* a function $g\colon \omega^k \to \omega$ ($f \leq_{\mathrm{E}} g$) if f can be generated from g and the initial elementary functions by the elementary definition schemes.
- Functions f and g are *equivalent* ($f \equiv_{\mathrm{E}} g$) if $f \leq_{\mathrm{E}} g$ and $g \leq_{\mathrm{E}} f$.

The elementary functions have nice closure properties, such as closure under bounded search and closure under bounded primitive recursion. These closure properties lead to useful alternative characterizations. To wit, the elementary functions are exactly the closure of 0, the successor function, the projection functions, the exponential function 2^x, and the max function under composition and bounded primitive recursion. One can also take advantage of the fact that Kleene's $\mathcal{T}$ predicate is elementary to show that the elementary functions are exactly those functions that can be computed by Turing machines that run in elementary time. That is, f is elementary if and only if there is a Turing machine computing f that runs in time $O(2^n_k)$ for some k, where 2_k is the k^{th} iterate of the exponential function (so $2^n_2 = 2^{2^n}$, $2^n_3 = 2^{2^{2^n}}$, and so forth). See [22, Chapter 1] for a presentation of the above-mentioned facts.

We study the class of all functions with elementary graphs, quasi-ordered by $\leq_{\mathrm{E}}$. By the discussion in [14, Section 1], it suffices to consider the so-called *honest* functions, as for every function f with elementary graph, there is an honest function g with $g \equiv_{\mathrm{E}} f$.

Definition 2.2. A function $f\colon \omega^n \to \omega$ is *honest* if

- f is unary: $n = 1$;
- f dominates 2^x: $\forall x(f(x) \geq 2^x)$;
- f is monotone: $\forall x(f(x) \leq f(x+1))$; and
- f has elementary graph.

The idea behind the terminology is that the output of an honest function gives some indication of how long the computation took. If f is honest, then there is a Turing machine computing f whose runtime is elementary in f. What would be considered *dishonest* is a Turing machine that makes long computations to produce short outputs (see, for example, [1]).

We can now define the honest elementary degrees.

Definition 2.3.

- The *honest elementary degree* of an honest function f is
$$\deg_{\mathrm{E}}(f) = \{g : g \text{ is honest and } g \equiv_{\mathrm{E}} f\}.$$
- The set of *honest elementary degrees* is $\mathcal{H} = \{\deg_{\mathrm{E}}(f) : f \text{ is honest}\}$.

The $\leq_{\mathrm{E}}$ relation induces a partial order on $\mathcal{H}$ in the usual way: for honest functions f and g, define $\deg_{\mathrm{E}}(f) \leq_{\mathrm{E}} \deg_{\mathrm{E}}(g)$ if $f \leq_{\mathrm{E}} g$. The resulting structure is a distributive lattice with join defined by $\deg_{\mathrm{E}}(f) \cup \deg_{\mathrm{E}}(g) = \deg_{\mathrm{E}}(\max[f,g])$ and meet defined by $\deg_{\mathrm{E}}(f) \cap \deg_{\mathrm{E}}(g) = \deg_{\mathrm{E}}(\min[f,g])$, and this lattice has a minimum element $\mathbf{0} = \deg_{\mathrm{E}}(2^x)$ (see [16]). Here $\max[f,g]$ is the function defined by $\max[f,g](x) = \max(f(x), g(x))$, and the function $\min[f,g]$ is defined analogously.

For a function $f : \omega \to \omega$ and a $k \in \omega$, let f^k denote the k^{th} iterate of f, defined by $f^0(x) = x$ and $f^{k+1}(x) = f(f^k(x))$. For functions $f, g\colon \omega \to \omega$, write $f \leq g$ if g dominates f: $\forall x(f(x) \leq g(x))$. Kristiansen's growth theorem ([11]; see [16, Theorem 2.3]) characterizing the $\leq_{\mathrm{E}}$ relation on honest functions in terms of rates of growth is the key tool for working with the honest elementary degrees.

Growth theorem ([11])**.** *If f and g are honest functions, then $f \leq_{\mathrm{E}} g$ if and only if $f \leq g^k$ for some $k \in \omega$.*

3. Honest elementary degrees without the cupping property

Kristiansen's result [14, Theorem 3.4] (see also [17, Theorem 10] and [16, Theorem 5.3]) states that every honest elementary degree that is sufficiently large has the cupping property, where 'sufficiently large' is made precise by the following definition.

Definition 3.1.

- For functions $f, g\colon \omega \to \omega$, define $f \ll_{\mathrm{E}} g$ if some fixed iterate of g eventually dominates every iterate of f: $(\exists k)(\forall m)(\forall^\infty x)(f^m(x) \leq g^k(x))$.
- For honest elementary degrees $\deg_{\mathrm{E}}(f)$ and $\deg_{\mathrm{E}}(g)$, define $\deg_{\mathrm{E}}(f) \ll_{\mathrm{E}} \deg(g)$ if $f \ll_{\mathrm{E}} g$.

(The notation '$(\forall^\infty x)\varphi(x)$' abbreviates '$\exists n \forall x(x > n \to \varphi(x))$,' which means that $\varphi(x)$ holds for almost every x.)

As an honest function is equivalent to its finite iterations, it is easy to see that $\mathbf{a} \ll_{\mathrm{E}} \mathbf{b}$ if and only if there is a $g \in \mathbf{b}$ that eventually dominates every $f \in \mathbf{a}$. We refer the reader to [16, 17] for more information concerning the $\ll_{\mathrm{E}}$ relation, including its original definition in terms of universal functions. We remark that although $<_{\mathrm{E}}$ is a dense partial ordering of $\mathcal{H}$ by work of Machtey [20] (see also [14, 16]), it is not known whether $\ll_{\mathrm{E}}$ is a dense partial ordering of $\mathcal{H}$ (see [21, Section 4]). The precise statement of Kristiansen's theorem on cupping is the following.

Theorem 3.2 ([14, Theorem 3.4])**.** *If* $\mathbf{a}$ *and* $\mathbf{b}$ *are honest elementary degrees with* $\mathbf{0} \ll_{\mathrm{E}} \mathbf{a} <_{\mathrm{E}} \mathbf{b}$*, then* $\mathbf{a}$ *cups to* $\mathbf{b}$*.*

Thus if $\mathbf{a}$ is an honest elementary degree with $\mathbf{a} \gg_{\mathrm{E}} \mathbf{0}$, then $\mathbf{a}$ has the cupping property. On the other hand, $\mathbf{0}$, being the minimum degree, certainly does not have the cupping property. Kristiansen, Schlage-Puchta, and Weiermann [17] (and again Kristiansen, Lubarsky, Schlage-Puchta, and Weiermann [16]) therefore ask if Theorem 3.2 can be improved to all $\mathbf{a} >_{\mathrm{E}} \mathbf{0}$. We prove that this is not the case.

Our technical theorem says that if $\mathbf{b} \gg_{\mathrm{E}} \mathbf{0}$, then there is a $\mathbf{a} >_{\mathrm{E}} \mathbf{0}$ that can only cup to degrees $\geq_{\mathrm{E}} \mathbf{b}$ via degrees that are already $\geq_{\mathrm{E}} \mathbf{b}$. Once we have this theorem, it is easy to produce a non-zero $\mathbf{a} <_{\mathrm{E}} \mathbf{b}$ that does not cup to $\mathbf{b}$ by appealing to the distributive lattice structure of $\mathcal{H}$. Thus we see that $\mathbf{b}$ has the anti-cupping property and that there is a non-zero $\mathbf{a}$ that does not have the cupping property.

Let $g \in \mathbf{b}$. To prove the theorem, we need to produce an honest $f >_{\mathrm{E}} 2^x$ such that for every honest h, if $\max[f, h] \geq_{\mathrm{E}} g$ then $h \geq_{\mathrm{E}} f$. Over the course of its computation, f keeps track of a set C of (indices of) functions h that look like they might satisfy $\max[f, h]^e \geq g$ for some e. Here $\max[f, h]^e$ is the e^{th} iterate of the function $\max[f, h]$. For each $h \in C$, f tries to stay below h so that if $\max[f, h]^e$ really is $\geq g$, then h will eventually dominate f. By removing h from C when learning that $\max[f, h]^e \not\geq g$, f can find safe numbers x for which $f(x)$ can be large in order to ensure that $f >_{\mathrm{E}} 2^x$.

Theorem 3.3. *For every* $\mathbf{b} \in \mathcal{H}$ *with* $\mathbf{b} \gg_{\mathrm{E}} \mathbf{0}$*, there is an* $\mathbf{a} \in \mathcal{H}$ *with* $\mathbf{a} >_{\mathrm{E}} \mathbf{0}$ *such that* $(\forall \mathbf{c} \in \mathcal{H})[(\mathbf{a} \cup \mathbf{c} \geq_{\mathrm{E}} \mathbf{b}) \rightarrow (\mathbf{c} \geq_{\mathrm{E}} \mathbf{b})]$.

Proof. For notational ease, we intentionally conflate a Turing machine with the partial function that it computes. Let $(\Phi_e : e \in \omega)$ be the usual effective list of all Turing machines. For each Φ_e, let $\widehat{\Phi}_e$ be the Turing machine that, on input n, runs Φ_e on inputs $0, 1, \ldots, n$ and, if all of these computations halt, outputs the maximum of 2^n and the total number of steps that the Φ_e computations took. The Turing machine $\widehat{\Phi}_e$ is essentially the *honest associate* of Φ_e as defined in [15], and we have that

- if $\widehat{\Phi}_e$ is total, then it is honest; and
- if h is honest, then there is an e such that $h \equiv_{\mathrm{E}} \widehat{\Phi}_e$ (see [15, Lemma 4]).

We also think of $\widehat{\Phi}_e(n)$ as being the number of steps in the computation of $\widehat{\Phi}_e(n)$ because the runtime of $\widehat{\Phi}_e$ is $O(\widehat{\Phi}_e)$.

Let Γ be a Turing machine computing a representative of $\mathbf{b}$ that, by the assumption $\mathbf{b} \gg_{\mathrm{E}} \mathbf{0}$, eventually dominates every elementary function. Define a Turing machine Ψ that behaves as follows on input n.

- Initialize $k \coloneqq 2$, $M \coloneqq 1$, and $C \coloneqq \{0\}$.
- Main loop: for each $m \leq n$, run $\widehat{\Phi}_e(m)$ for all $e \in C$ in a dovetailing fashion for at most 2^m_k steps each.
  - If some $\widehat{\Phi}_e(m)$ halts with output N:
    - $*$ For each $e \in C$ and each $\ell < m$:
      - $\cdot$ Run $\Gamma(\ell)$ and $\max[\Psi, \widehat{\Phi}_e]^e(\ell)$ for m steps each, aborting the computation of $\max[\Psi, \widehat{\Phi}_e]^e(\ell)$ if it produces numbers $\geq m$. (Observe that the value of $\Psi(\ell)$

is the value of M after iteration ℓ of the main loop, so the computation of $\max[\Psi, \widehat{\Phi}_e]^e(\ell)$ can be facilitated by storing the previous values of M in a table.)

· If $\Gamma(\ell)$ and $\max[\Psi, \widehat{\Phi}_e]^e(\ell)$ both halt within m steps and $\max[\Psi, \widehat{\Phi}_e]^e(\ell) < \Gamma(\ell)$, then set $C := C \setminus \{e\}$.

* Set $M := \max\{M, N, 2^m\}$.
* Stop running the $\widehat{\Phi}_e(m)$'s, and go to the next iteration of the main loop.

– Else:
  * Set $k := k + 1$.
  * Let i be the least number that has never been in C, and set $C := C \cup \{i\}$.
  * Set $M := \max\{M, 2^m_k\}$.

• Output M when the main loop terminates.

Claim 1. Ψ *is honest.*

Proof of claim. Clearly Ψ is unary. For a given input n, let M_m denote the value of M after iteration $m \leq n$ of the main loop. It is easy to see that $M_m \geq 2^m$, that M_m is monotonic in m, and that $M_m = \Psi(m)$. Thus Ψ dominates 2^x and is monotonic. We need to show that Ψ has elementary graph. Recall from the discussion following Definition 2.1 that the elementary functions are exactly the functions that can be computed in elementary time. Thus we need to show that the graph of Ψ is computable in elementary time. As the output of Ψ is always bigger than the corresponding input, it suffices to show that the runtime of $\Psi(n)$ is elementary in the value of $\Psi(n)$. In fact, we show that the runtime of Ψ is polynomial in its outputs.

Let C_m be the value of C at the beginning of iteration m of the main loop. Notice that at most one number is added to C during each iteration, so $|C_m| \leq m + 1$. In iteration m, either there is an $e \in C_m$ such that $\widehat{\Phi}_e(m)$ halts within 2^m_k steps, or there is not. Consider first the case in which there is an $e_0 \in C_m$ is such that $\widehat{\Phi}_{e_0}(m)$ halts within 2^m_k steps with output N. Then $\widehat{\Phi}_{e_0}(m)$ halts within $O(N)$ steps, so each $\widehat{\Phi}_e(m)$ with $e \in C_m$ is run for $O(N)$ steps because the dovetailed execution of these machines halts when $\widehat{\Phi}_{e_0}(m)$ halts. Therefore $O(|C_m|N)$ steps are spent running the $\widehat{\Phi}_e(m)$'s. Afterward, for each $e \in C$ and each $\ell < m$, $\Gamma(\ell)$ and $\max[\Psi, \widehat{\Phi}_e]^e(\ell)$ are run for at most m steps each and compared. This takes $O(|C_m|m^2)$ steps. Thus the total number of steps taken in this case is

$$O(|C_m|N + |C_m|m^2) = O(mN + m^3) = O(M_n^3) = O(\Psi(n)^3),$$

where the first equality is because $|C_m| \leq m + 1$, and the second equality is because $m \leq 2^m \leq M_m \leq M_n$ and, in this case, $N \leq M_m \leq M_n$.

Now consider the case in which no $\widehat{\Phi}_e(m)$ halts within 2^m_k steps. In this case, $O(|C_m|2^m_k)$ steps are spent running the $\widehat{\Phi}_e(m)$'s. Thus the total number of steps taken is $O(M_n^2) = O(\Psi(n)^2)$ because, in this case, $|C_m| \leq m + 1 \leq 2^m_k \leq M_m \leq M_n$.

Thus iteration m of the main loop takes $O(\Psi(n)^3)$ steps. So Ψ runs in time $O(\Psi(n)^4)$. □

Claim 2. $\Psi >_{\mathrm{E}} 2^x$.

Proof of claim. It suffices to show that k increases infinitely often in the sense that for every n_0 there is an $n \geq n_0$ such that, in the execution of $\Psi(n)$, the value of k increases at the end of iteration n of the main loop. This is because $\Psi(n) \geq 2^n_k$ in this case, and, therefore, if k increases infinitely often, then $\forall k \exists n(\Psi(n) \geq 2^n_k)$. This implies that $\Psi >_{\mathrm{E}} 2^x$ by the growth theorem.

To show that k increases infinitely often, we show that for every n_0 there is an $n \geq n_0$ such that either k increases or $|C|$ decreases during iteration n of the main loop. Suppose for a contradiction that there is an n_0 such that k never increases and $|C|$ never decreases after iteration n_0. Then iteration n enters the 'if' case of the main loop for all $n \geq n_0$. This implies that there is a fixed k_0

such that, for all $n \geq n_0$, $\min\{\widehat{\Phi}_e(n) : e \in C\} < 2^n_{k_0}$ and $\Psi(n) = \max\{\Psi(n-1), N, 2^n\}$ for some $N < 2^n_{k_0}$. It follows that Ψ is elementary by the growth theorem. Furthermore, C never changes after iteration n_0 because no numbers are removed from C by assumption, and no numbers are added to C because the main loop never enters the 'else' case. Thus there must be an e in this fixed C such that $\widehat{\Phi}_e \not\gg_{\mathrm{E}} 0$ because there must be an $e \in C$ for which $\widehat{\Phi}_e(n) < 2^n_{k_0}$ holds for infinitely many n. For this e, $\max[\Psi, \widehat{\Phi}_e]^e \equiv_{\mathrm{E}} \widehat{\Phi}_e \not\gg_{\mathrm{E}} 0$, so, because $\Gamma \gg_{\mathrm{E}} 0$, there is an ℓ such that $\max[\Psi, \widehat{\Phi}_e]^e(\ell) < \Gamma(\ell)$. Therefore e is removed from C during iteration n of the main loop once n is large enough so that the computation witnessing that $\max[\Psi, \widehat{\Phi}_e]^e(\ell) < \Gamma(\ell)$ takes at most n steps. This contradicts that $|C|$ never decreases after iteration n_0.

Now it is easy to see that k increases infinitely often. If not, there is an n_0 such that k never increases after iteration n_0. In this case, by the preceding argument, $|C|$ can only decrease, so there is an $n \geq n_0$ such that C is empty at the start of iteration n. In this situation, the main loop enters the 'else' case, and k is increased, contradicting that k never increases. $\square$

Claim 3. *For every honest h, either $\max[\Psi, h] \not\geq_{\mathrm{E}} \Gamma$ or $h \geq_{\mathrm{E}} \Psi$.*

Proof of claim. First, consider an index e of a total $\widehat{\Phi}_e$. From the proof of the previous claim, k increases infinitely often, which implies that e is eventually added to C. If e is never removed from C, then Ψ is $O(\max[\widehat{\Phi}_e, 2^x])$, which implies that $\widehat{\Phi}_e \geq_{\mathrm{E}} \Psi$. On the other hand, if e is eventually removed from C, then there is an ℓ such that $\max[\Psi, \widehat{\Phi}_e]^e(\ell) < \Gamma(\ell)$.

Now consider an honest h and the infinitely many indices e such that $\widehat{\Phi}_e \equiv_{\mathrm{E}} h$ and $\forall x(h(x) \leq \widehat{\Phi}_e(x))$. If one such e enters C and is never removed, then $h \equiv_{\mathrm{E}} \widehat{\Phi}_e \geq_{\mathrm{E}} \Psi$. If every such e is eventually removed from C after it enters, then for infinitely many e there is an ℓ such that $\max[\Psi, h]^e(\ell) \leq \max[\Psi, \widehat{\Phi}_e]^e(\ell) < \Gamma(\ell)$. Hence $\max[\Psi, h] \not\geq_{\mathrm{E}} \Gamma$ by the growth theorem. $\square$

Let $\mathbf{a} = \deg_{\mathrm{E}}(\Psi)$. Then $\mathbf{a} \in \mathcal{H}$ by Claim 1, and $\mathbf{a} >_{\mathrm{E}} \mathbf{0}$ by Claim 2. If $\mathbf{c} \in \mathcal{H}$ is such that $\mathbf{a} \cup \mathbf{c} \geq_{\mathrm{E}} \mathbf{b}$, then $\mathbf{c} \geq_{\mathrm{E}} \mathbf{a}$ by Claim 3. Therefore $\mathbf{c} \geq_{\mathrm{E}} \mathbf{a} \cup \mathbf{c} \geq_{\mathrm{E}} \mathbf{b}$ as desired, which completes the proof. $\square$

Corollary 3.4. *Every $\mathbf{b} \in \mathcal{H}$ with $\mathbf{b} \gg_{\mathrm{E}} \mathbf{0}$ has the anti-cupping property.*

Proof. Given $\mathbf{b} \gg_{\mathrm{E}} \mathbf{0}$, by Theorem 3.3, let $\mathbf{x} >_{\mathrm{E}} \mathbf{0}$ be such that $(\forall \mathbf{c} \in \mathcal{H})[(\mathbf{x} \cup \mathbf{c} \geq_{\mathrm{E}} \mathbf{b}) \to (\mathbf{c} \geq_{\mathrm{E}} \mathbf{b})]$. Let $\mathbf{a} = \mathbf{x} \cap \mathbf{b}$. One readily checks that $\mathbf{b} \gg_{\mathrm{E}} \mathbf{0}$ and $\mathbf{x} >_{\mathrm{E}} \mathbf{0}$ imply that $\mathbf{a} >_{\mathrm{E}} \mathbf{0}$. Now consider a $\mathbf{c} \in \mathcal{H}$ such that $\mathbf{a} \cup \mathbf{c} = \mathbf{b}$. Clearly $\mathbf{c} \leq_{\mathrm{E}} \mathbf{b}$. On the other hand, using the fact that $\mathcal{H}$ is a distributive lattice,

$$\mathbf{b} = \mathbf{a} \cup \mathbf{c} = (\mathbf{x} \cap \mathbf{b}) \cup \mathbf{c} = (\mathbf{x} \cup \mathbf{c}) \cap (\mathbf{b} \cup \mathbf{c}) = (\mathbf{x} \cup \mathbf{c}) \cap \mathbf{b}.$$

Thus $\mathbf{x} \cup \mathbf{c} \geq_{\mathrm{E}} \mathbf{b}$, which implies that $\mathbf{c} \geq_{\mathrm{E}} \mathbf{b}$ by the choice of $\mathbf{x}$. Thus $\mathbf{c} = \mathbf{b}$. This shows that $\mathbf{a}$ does not cup to $\mathbf{b}$, so $\mathbf{a}$ witnesses that $\mathbf{b}$ has the anti-cupping property. $\square$

Thus, for every $\mathbf{b} \gg_{\mathrm{E}} \mathbf{0}$, there is a non-zero $\mathbf{a} <_{\mathrm{E}} \mathbf{b}$ that does not have the cupping property as witnessed by $\mathbf{b}$. One could also prove Corollary 3.4 by directly ensuring that $\mathbf{a} \leq_{\mathrm{E}} \mathbf{b}$ in the proof of Theorem 3.3 (by enforcing that Γ dominates Ψ).

Question 3.5. In Theorem 3.3 and Corollary 3.4, can $\mathbf{b} \gg_{\mathrm{E}} \mathbf{0}$ be weakened to $\mathbf{b} >_{\mathrm{E}} \mathbf{0}$?

We have shown that there are non-zero honest elementary degrees that do not have the cupping property. For the curious readier, we briefly summarize what is known about *capping* in $\mathcal{H}$. An element a of a lattice *caps* to $b < a$ if there is a $c > b$ such that $a \cap c = b$. If $\mathbf{a}, \mathbf{b} \in \mathcal{H}$ are such that $\mathbf{b} \ll_{\mathrm{E}} \mathbf{a}$, then $\mathbf{a}$ does not cap to $\mathbf{b}$ [13]. However, this result does not provide a characterization of capping in $\mathcal{H}$ because there are $\mathbf{a}, \mathbf{b} \in \mathcal{H}$ with $\mathbf{b} <_{\mathrm{E}} \mathbf{a}$ and $\mathbf{b} \not\ll_{\mathrm{E}} \mathbf{a}$ such that $\mathbf{a}$ does not cap to $\mathbf{b}$ [16].

4. Degrees of relative provability and honest α-elementary degrees

In this section, we provide a basic introduction to the theory of the degrees of relative provability and the honest α-elementary degrees. We assume familiarity with Peano arithmetic and its fragments. The most important fragment for us is $\mathrm{I}\Sigma_1$, which consists of the basic axioms and the induction scheme

$$[\varphi(0) \wedge \forall n(\varphi(n) \to \varphi(n+1))] \to \forall n \varphi(n)$$

restricted to Σ_1 formulas φ. $\mathrm{I}\Sigma_1$ is Σ_1-*complete*, meaning that $\mathrm{I}\Sigma_1$ proves every true Σ_1 sentence. $\mathrm{I}\Sigma_1$ also suffices to define the Π_1 truth predicate 'true(n),' which states that n codes (i.e., is the Gödel number of) a true Π_1 sentence (see, for example, [10, Section I.1(d)] for details). We note that the Π_1 truth predicate is itself Π_1. Finally, we mention that $\mathrm{I}\Sigma_1$ proves $\mathrm{B}\Sigma_1$, which consists of the basic axioms, the induction scheme restricted to Σ_0 formulas, and the bounding scheme

$$\forall a[(\forall n < a)(\exists m)\varphi(n,m) \to (\exists b)(\forall n < a)(\exists m < b)\varphi(n,m)]$$

restricted to Σ_1 formulas φ (see [10, Theorem 2.5]). Throughout this section, every theory is assumed to be in the language of arithmetic, to be consistent, and to extend $\mathrm{I}\Sigma_1$.

First we describe Cai's degrees of relative provability [7].

Definition 4.1. Fix a theory T. For a Turing machine Φ, let $\mathrm{tot}(\Phi)$ be the Π_2 sentence expressing that Φ is total.

- Turing machine Φ *provably reduces* to Turing machine Ψ ($\Phi \leq_T \Psi$) if $T \vdash \mathrm{tot}(\Psi) \to \mathrm{tot}(\Phi)$.
- Turing machines Φ and Ψ are *provably equivalent* ($\Phi \equiv_T \Psi$) if $\Phi \leq_T \Psi$ and $\Psi \leq_T \Phi$.
- The *provability degree* of a Turing machine Φ is

$$\deg_T(\Phi) = \{\Psi : \Psi \text{ is a Turing machine and } \Psi \equiv_T \Phi\}$$

- The set of *provability degrees* is $\mathcal{P}_T = \{\deg_T(\Phi) : \Phi \text{ is a total Turing machine}\}$.

We follow the usual convention that a Turing machine halts on input n if and only if it halts on all inputs $m \leq n$. This is without loss of generality, assuming $\mathrm{I}\Sigma_1$. For a Turing machine Φ, let $\widehat{\Phi}$ be the Turing machine that, on input n, runs $\Phi(0), \Phi(1), \dots, \Phi(n)$ in succession and halts if and only if $(\forall m \leq n)(\Phi(m)\downarrow)$. Then $T \vdash \mathrm{tot}(\widehat{\Phi}) \leftrightarrow \mathrm{tot}(\Phi)$.

It is easy to see that $\leq_T$ quasi-orders the Turing machines and therefore induces a partial order on $\mathcal{P}_T$. In fact, $\mathcal{P}_T$ is a distributive lattice. Let Φ and Ψ be two total Turing machines. Then $\deg_T(\Phi) \cup \deg_T(\Psi) = \deg_T(\Gamma)$, where, for each n, $\Gamma(n)$ runs $\Phi(n)$ and $\Psi(n)$ simultaneously and halts when both $\Phi(n)$ and $\Psi(n)$ halt. Similarly, $\deg_T(\Phi) \cap \deg_T(\Psi) = \deg_T(\Theta)$, where, for each n, $\Theta(n)$ runs $\Phi(n)$ and $\Psi(n)$ simultaneously and halts when either $\Phi(n)$ halts or $\Psi(n)$ halts. Notice that $T \vdash \mathrm{tot}(\Gamma) \leftrightarrow (\mathrm{tot}(\Phi) \wedge \mathrm{tot}(\Psi))$ and that $T \vdash \mathrm{tot}(\Theta) \leftrightarrow (\mathrm{tot}(\Phi) \vee \mathrm{tot}(\Psi))$. $\mathcal{P}_T$ also has a minimum element $\mathbf{0}$, which is the degree of any Turing machine that T proves is total, such as the machine that immediately halts and outputs 0 on every input. See [7] for proofs of these facts.

We remark that if φ is a true Π_2 sentence, then there is a total Turing machine Φ such that $T \vdash \varphi \leftrightarrow \mathrm{tot}(\Phi)$. Thus one may think of $\mathcal{P}_T$ as the Lindenbaum algebra of T restricted to true Π_2 sentences.

Now we describe Kristiansen, Schlage-Puchta, and Weiermann's honest α-elementary degrees [17]. First, we recall that every ordinal $\alpha < \epsilon_0$ has a Cantor normal form $\omega^{\alpha_0} + \omega^{\alpha_1} + \cdots + \omega^{\alpha_{n-1}} + 0$, where the ordinals $\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_{n-1}$ are themselves in Cantor normal form. This allows us to define the *norm* of an ordinal $\alpha < \epsilon_0$ by induction on its Cantor normal form.

Definition 4.2. Let $\alpha < \epsilon_0$. The *norm* of α, $N(\alpha)$, is defined by induction on α's Cantor normal form by $N(0) = 0$, $N(\beta + \gamma) = N(\beta) + N(\gamma)$, and $N(\omega^\beta) = 1 + N(\beta)$.

This definition of norm allows us to make sense of iterating a function transfinitely many times.

Definition 4.3. Let $f\colon \omega \to \omega$, and let $\alpha < \epsilon_0$. The α^{th} iterate of f, f_α, is defined inductively by

$$\begin{aligned} f_0(n) &= f(n) \\ f_\alpha(n) &= \max\{f_\beta(f_\beta(n)) : (\beta < \alpha) \wedge (N(\beta) \leq N(\alpha) + n)\} \qquad \text{for } \alpha > 0. \end{aligned}$$

We now define the 'α-elementary in' relation by adding transfinite iteration to the elementary definition schemes of Definition 2.1.

Definition 4.4. Let $\alpha \leq \epsilon_0$.

- A function $f\colon \omega^n \to \omega$ is *α-elementary in* a function $g\colon \omega^k \to \omega$ ($f \leq_{\alpha\mathrm{E}} g$) if f can be generated from g and the initial elementary functions of Definition 2.1 by the elementary definition schemes of Definition 2.1 and by β-iteration for all $\beta < \alpha$.
- Functions f and g are *equivalent* ($f \equiv_{\alpha\mathrm{E}} g$) if $f \leq_{\alpha\mathrm{E}} g$ and $g \leq_{\alpha\mathrm{E}} f$.

We note that if f is honest and $\alpha < \epsilon_0$, then f_α is also honest [17].

Let $\mathrm{SLim} = \{\alpha \leq \epsilon_0 : (\exists \beta > 0)(\alpha = \omega^\beta)\}$. For $\alpha \in \mathrm{SLim}$, Kristiansen, Schlage-Puchta, and Weiermann give the following generalization of the growth theorem.

Generalized growth theorem ([17])**.** *If $\alpha \in \mathrm{SLim}$ and f and g are honest functions, then $f \leq_{\alpha\mathrm{E}} g$ if and only if $f \leq g_\beta$ for some $\beta < \alpha$.*

We define the honest α-elementary degrees for $\alpha \in \mathrm{SLim}$ analogously to Definition 2.3.

Definition 4.5. Let $\alpha \in \mathrm{SLim}$.

- The *honest α-elementary degree* of an honest function f is
$$\deg_{\alpha\mathrm{E}}(f) = \{g : g \text{ is honest and } g \equiv_{\alpha\mathrm{E}} f\}$$
- The set of *honest α-elementary degrees* is $\mathcal{H}_\alpha = \{\deg_{\alpha\mathrm{E}}(f) : f \text{ is honest}\}$.

Again, $\mathcal{H}_\alpha$ is a distributive lattice with partial order induced by $\leq_{\alpha\mathrm{E}}$, join and meet defined via max and min as with $\mathcal{H}$, and minimum element $\mathbf{0} = \deg_{\alpha\mathrm{E}}(2^x)$ [17]. Notice that $\mathcal{H}_\omega = \mathcal{H}$ because the finite iterates of a function can be defined using the elementary definition schemes.

For a theory T, let T^+ denote T extended by all true Π_1 sentences. The connection between the degrees of relative provability and the honest α-elementary degrees is made clear by the following result of Kristiansen.

Theorem 4.6 ([15])**.** *$\mathcal{P}_{\mathrm{PA}^+}$ and $\mathcal{H}_{\epsilon_0}$ are isomorphic.*

$\mathcal{P}_{T^+}$ and $\mathcal{H}_\alpha$ should also be isomorphic for various fragments T of PA and the appropriate ordinals α, but the details still need to be checked. However, we do not know which, if any, of the $\mathcal{P}_{T^+}$'s are isomorphic and which, if any, of the $\mathcal{H}_\alpha$'s are isomorphic.

Question 4.7. Are the $\mathcal{P}_{T^+}$'s isomorphic for the fragments T of PA extending $\mathrm{I}\Sigma_1$? Are they elementarily equivalent? Are the $\mathcal{H}_\alpha$'s isomorphic for the $\alpha \in \mathrm{SLim}$? Are they elementarily equivalent?

In the following section, we extend a result of Cai's implying that there are non-zero elements of $\mathcal{P}_{T^+}$ that do not have the cupping property [8]. Thus, by Theorem 4.6, there are non-zero elements of $\mathcal{H}_{\epsilon_0}$ that do not have the cupping property. We can also prove this fact directly by running the proof of Theorem 3.3 in the context of $\mathcal{H}_{\epsilon_0}$. In fact, the proof of Theorem 3.3 can be modified to show that, for every $\alpha \in \mathrm{SLim}$, sufficiently large elements of $\mathcal{H}_\alpha$ have the anti-cupping property.

Fix $\alpha \in \mathrm{SLim}$, and reinterpret '$\ll$' in the context of $\mathcal{H}_\alpha$ by defining $f \ll_{\alpha\mathrm{E}} g$ to mean that there is a $\gamma < \alpha$ such that g_γ eventually dominates f_β for every $\beta < \alpha$: $(\exists \gamma < \alpha)(\forall \beta < \alpha)(\forall^\infty x)(f_\beta(x) \leq g_\gamma(x))$. In particular, $\mathbf{b} \gg_{\alpha\mathrm{E}} \mathbf{0}$ means that there is a $g \in \mathbf{b}$ that eventually dominates 2^x_β (the β^{th} iterate of 2^x) for every $\beta < \alpha$.

As in the proof of Theorem 3.3, fix a Turing machine Γ computing a member of $\mathbf{b}$ that eventually dominates 2^x_β for every $\beta < \alpha$. Fix an elementary *fundamental sequence* of ordinals $\alpha_0 < \alpha_1 < \dots$ that converges to α. That is, fix an elementary function that maps k to a code for α_k. Define the Turing machine Ψ as before, except now replace 2^m_k by $2^m_{\alpha_k}$ and replace $\max[\Psi, \widehat{\Phi}_e]^e$ by $\max[\Psi, \widehat{\Phi}_e]_{\alpha_e}$. Ψ is again honest because its runtime is elementary in its outputs. In order to honestly compute the $2^m_{\alpha_k}$'s, use the fact that if f is honest, then the predicate "$f_{\alpha_k}(x) = y$" is elementary, which is proven in the course of the proof of [17, Lemma 17]. Again, $\Psi >_{\alpha\mathrm{E}} 2^x$ because k must increase infinitely often. That k increases infinitely often implies that Ψ is not dominated by $2^x_{\alpha_k}$ for any k, which, as $\lim_k \alpha_k = \alpha$, implies that Ψ is not dominated by 2^x_β for any $\beta < \alpha$. Finally, for any honest h, either $\max[\Psi, h] \not\geq_{\alpha\mathrm{E}} \Gamma$ or $h \geq_{\alpha\mathrm{E}} \Psi$ by again considering the indices e for which $\widehat{\Phi}_e \equiv_{\alpha\mathrm{E}} h$ and whether or not they are all eventually removed from C.

5. Degrees of relative provability without the cupping property

Throughout this section, we assume that all theories considered are consistent theories in the language of arithmetic. In [8], Cai proves that if T is a recursively axiomatizable extension of PA, then there are non-zero elements of $\mathcal{P}_{T^+}$ that do not have the cupping property. We modify Cai's proof in a few ways that we hope will be helpful in future work comparing the degrees of relative provability to the honest α-elementary degrees.

First, we prove a stronger statement: every non-zero $\mathbf{b} \in \mathcal{P}_{T^+}$ has the anti-cupping property. Second, we work with $\mathcal{P}_{T^+}$ directly. Cai considers many different sub-algebras of T's Lindenbaum algebra, and he proves that, in the sub-algebra of true Π_1 sentences, there are non-zero elements that do not have the cupping property. He then obtains the corresponding result for $\mathcal{P}_{T^+}$ by relativization and an application of an isomorphism theorem. We use Cai's same strategy to directly define a total Turing machine Ψ whose degree does not non-trivially cup above a given non-zero degree $\mathbf{b}$. This construction is slightly more complicated than Cai's original construction, but we believe it has some benefits in addition to being technically interesting in its own right. The direct construction is easier to see in terms of Kristiansen's isomorphism from Theorem 4.6, and we hope that it will help decide whether or not $\mathbf{b} \gg_{\alpha\mathrm{E}} \mathbf{0}$ can be replaced by $\mathbf{b} >_{\alpha\mathrm{E}} \mathbf{0}$ in the $\mathcal{H}_\alpha$ cases. The direct construction also makes it a little easier to keep track of how much of T is being used. In our proof, we only assume that T extends $\mathrm{I}\Sigma_1$, whereas Cai assumes that T extends PA (though for Cai this assumption is mostly a matter of convenience).

We say that a number t *witnesses* a Σ_1 sentence $\exists n \varphi(n)$ if $(\exists n < t)\varphi(n)$. That is, t witnesses $\exists n\varphi(n)$ if t is large enough to verify that $\exists n\varphi(n)$ is true. Let $\ulcorner\varphi\urcorner$ denote the code of the formula φ according to some fixed Gödel numbering. We first define a helpful family of auxiliary Turing machines. For a Π_1 sentence η and a finite set C of pairs of the form $\langle\ulcorner\pi\urcorner, e\rangle$, where π is a Π_1 sentence and e is the index of a Turing machine, let $A^{\ulcorner\eta\urcorner}_C$ be the Turing machine that behaves as follows on input s.

- Initialize $t := s$.
- While `true`:
  - If t witnesses $\neg\eta$: halt and output 0.
  - If there is a $\langle\ulcorner\pi\urcorner, e\rangle \in C$ such that $(\forall n \leq s)(\Phi_{e,t}(n){\downarrow})$ and t does not witness $\neg\pi$: halt and output 0.
  - Else: set $t := t + 1$.

(The notation '$\Phi_{e,t}(n){\downarrow}$' means that the execution of $\Phi_e(n)$ halts within t steps.)

Many of the following arguments combine reasoning in ordinary mathematics with reasoning inside of a formal theory. We warn the reader that, to keep notational clutter to a minimum, we intentionally conflate the number $s \in \omega$ with the standard term that names it. For example, if we have determined that the Turing machine Φ halts on all inputs $n \leq s$ and then want to reason formally about this, we write '$(\forall n \leq s)(\Phi(n)\downarrow)$' instead of the more technically correct '$(\forall n \leq \overline{s})(\Phi(n)\downarrow)$,' where $\overline{s}$ is the name for s.

Lemma 5.1. *Let T be a recursively axiomatizable extension of* $\mathrm{I}\Sigma_1$.

(1)

$$T \vdash (\forall C)(\forall \langle \ulcorner \pi \urcorner, e\rangle \in C)[(\mathrm{true}(\ulcorner \pi \urcorner) \wedge \mathrm{tot}(\Phi_e)) \to \mathrm{tot}(A_C^{\ulcorner \eta \urcorner})].$$

(2) *Let η be a true Π_1 sentence, and let C be a finite set of pairs. Then*

$$T^+ \vdash \mathrm{tot}(A_C^{\ulcorner \eta \urcorner}) \leftrightarrow \bigvee_{\substack{\langle \ulcorner \pi \urcorner, e\rangle \in C \\ \pi \text{ is true}}} \mathrm{tot}(\Phi_e)$$

(*where the empty disjunction is considered to be false*).

Proof. First we prove (1). Working in T, suppose that $\langle \ulcorner \pi \urcorner, e\rangle \in C$, $\mathrm{true}(\ulcorner \pi \urcorner)$, and $\mathrm{tot}(\Phi_e)$. Consider an arbitrary s. By $\mathrm{tot}(\Phi_e)$ and $\mathrm{B}\Sigma_1$ (which is provable in $\mathrm{I}\Sigma_1$; see the discussion at the beginning of Section 4), there is a $t \geq s$ such that $(\forall n \leq s)(\Phi_{e,t}(n)\downarrow)$. Such a t does not witness $\neg\pi$ by the assumption $\mathrm{true}(\ulcorner \pi \urcorner)$. Therefore there is a least $t \geq s$ such that either t witnesses $\neg\eta$ or there is a $\langle \ulcorner \pi' \urcorner, e'\rangle \in C$ such that $(\forall n \leq s)(\Phi_{e',t}(n)\downarrow)$ and t does not witness $\neg\pi'$. Therefore $A_C^{\ulcorner \eta \urcorner}(s)\downarrow$. As s is arbitrary, we conclude $\mathrm{tot}(A_C^{\ulcorner \eta \urcorner})$.

The preceding argument also proves the '$\leftarrow$' direction of (2) because if π is true, then $\mathrm{true}(\ulcorner \pi \urcorner)$ is a true Π_1 sentence and hence an axiom of T^+. So we need to prove the '$\to$' direction of (2). We prove the contrapositive. Work in T^+ and suppose that

$$\bigwedge_{\substack{\langle \ulcorner \pi \urcorner, e\rangle \in C \\ \pi \text{ is true}}} \neg\, \mathrm{tot}(\Phi_e).$$

For the conjunct indexed by $\langle \ulcorner \pi \urcorner, e\rangle$, let s_e be such that $\forall t(\Phi_{e,t}(s_e)\uparrow)$. Let s be larger than all of the s_e's and large enough to witness $\neg\pi$ for all $\langle \ulcorner \pi \urcorner, e\rangle \in C$ with π false. Note that no t can witness $\neg\eta$ because η is assumed to be true, and thus $\mathrm{true}(\ulcorner \eta \urcorner)$ is an axiom of T^+. Thus $A_C^{\ulcorner \eta \urcorner}(s)\downarrow$ leads to a contradiction, so we must have that $A_C^{\ulcorner \eta \urcorner}(s)\uparrow$. Therefore $\neg\, \mathrm{tot}(A_C^{\ulcorner \eta \urcorner})$. □

The next theorem is our modification of Cai's [8, Theorem 7.1] and an analog of Theorem 3.3.

Theorem 5.2. *Let T be a recursively axiomatizable extension of* $\mathrm{I}\Sigma_1$. *For every $\mathbf{b} \in \mathcal{P}_{T^+}$ with $\mathbf{b} >_{T^+} \mathbf{0}$, there is an $\mathbf{a} \in \mathcal{P}_{T^+}$ such that $\mathbf{a} \cap \mathbf{b} >_{T^+} \mathbf{0}$ and $(\forall \mathbf{c} \in \mathcal{P}_{T^+})[(\mathbf{a} \cup \mathbf{c} \geq_{T^+} \mathbf{b}) \to (\mathbf{c} \geq_{T^+} \mathbf{b})]$.*

Proof. Let Γ be a total Turing machine with $\deg_{T^+}(\Gamma) = \mathbf{b} >_{T^+} \mathbf{0}$. Define a Turing machine Ψ that behaves as follows on input s. The definition of Ψ uses the recursion theorem (see [23, Theorem II.3.1]) to assume that Ψ has access to its own code.

- Initialize `runA` $:=$ `false`, $C := \emptyset$, $p := 0$, $\ulcorner \eta \urcorner := \ulcorner 0 = 0 \urcorner$.
- Main loop: for each $m \leq s$, do the following:
  - If `runA` is `true` and m does not witness $\neg\eta$: execute $A_C^{\ulcorner \eta \urcorner}(m)$.
  - Else:
    - Set `runA` $:=$ `false`.
    - If p codes a proof witnessing

      $$T + \pi \vdash (\mathrm{tot}(\Phi_e) \wedge \mathrm{tot}(\Psi)) \to \mathrm{tot}(\Gamma)$$

      for some Π_1 sentence π and some e: set $C := C \cup \{\langle \ulcorner \pi \urcorner, e\rangle\}$.
    - If p codes a proof witnessing

      $$T + \eta' \vdash \mathrm{tot}(\Psi) \vee \mathrm{tot}(\Gamma)$$

      for some Π_1 sentence η': set $\ulcorner \eta \urcorner := \ulcorner \eta' \urcorner$, and set `runA` $:=$ `true`.

  * Set $p \coloneqq p+1$.
- Output 0.

Claim 1. Ψ *is total.*

Proof of claim. Suppose for a contradiction that $\Psi(s)\uparrow$ for some s. Observe that $\Psi(s)\uparrow$ is a true Π_1 sentence and hence an axiom of T^+. Thus $T^+ \vdash \Psi(s)\uparrow$, so $T^+ \vdash \neg\,\mathrm{tot}(\Psi)$ (in fact, T^+ proves all true Σ_2 sentences by the same argument). As $\Psi(s)\uparrow$, it must be that the execution of $\Psi(s)$ executes $A_C^{\ulcorner\eta\urcorner}(m)$ for some η, C, and m for which $A_C^{\ulcorner\eta\urcorner}(m)\uparrow$. For this to happen, it must be that $T+\eta \vdash \mathrm{tot}(\Psi)\vee\mathrm{tot}(\Gamma)$. Furthermore, it is easy to see that $A_C^{\ulcorner\eta\urcorner}$ is total if η is false. So it must be that η is true, in which case $T^+ \vdash \mathrm{tot}(\Psi)\vee\mathrm{tot}(\Gamma)$. From $T^+ \vdash \neg\,\mathrm{tot}(\Psi)$ and $T^+ \vdash \mathrm{tot}(\Psi)\vee\mathrm{tot}(\Gamma)$ we conclude that $T^+ \vdash \mathrm{tot}(\Gamma)$, which contradicts that $\deg_{T^+}(\Gamma) = \mathbf{b} >_{T^+} \mathbf{0}$. ☐

Claim 2. $T^+ \nvdash \mathrm{tot}(\Psi)\vee\mathrm{tot}(\Gamma)$.

Proof of claim. Suppose for a contradiction that $T^+ \vdash \mathrm{tot}(\Psi)\vee\mathrm{tot}(\Gamma)$, and let p_0 be the least number coding a proof witnessing that $T+\eta_0 \vdash \mathrm{tot}(\Psi)\vee\mathrm{tot}(\Gamma)$ for some true Π_1 sentence η_0. Then if $p < p_0$ codes a proof witnessing that $T+\eta \vdash \mathrm{tot}(\Psi)\vee\mathrm{tot}(\Gamma)$ for some Π_1 sentence η, this η must be false. Therefore, if the main loop is iterated enough times, p is eventually set to p_0, in which case $\ulcorner\eta\urcorner$ is set to $\ulcorner\eta_0\urcorner$. Let s_0 be the least number such that p is set to p_0 during the execution of $\Psi(s_0)$. As Ψ is total by Claim 1, we see that s_0 satisfies the Σ_1 formula $\varphi(s)$ saying "there is a sequence $(r_i)_{i\le s}$ such that $(\forall i \le s)(r_i$ codes a halting run of $\Psi(i))$, p is set to p_0 during run r_s, and $(\forall i < s)(p$ is never set to p_0 during run $r_i)$." Thus $\varphi(s_0)$ is a true Σ_1 sentence, so $T \vdash \varphi(s_0)$ by Σ_1-completeness. That is, T proves that s_0 is indeed the least number such that p is set to p_0 (and also $\ulcorner\eta\urcorner$ is set to $\ulcorner\eta_0\urcorner$) during the execution of $\Psi(s_0)$. Let C_0 be the value of C when p is set to p_0.

Now we work in T^+ to show that $T^+ \vdash (\forall s > s_0)(\Psi(s)\downarrow) \leftrightarrow (\forall s > s_0)(A_{C_0}^{\ulcorner\eta_0\urcorner}(s)\downarrow)$. First suppose that $(\forall s > s_0)(\Psi(s)\downarrow)$. Consider the execution of $\Psi(s)$ for an $s > s_0$. We know that p is set to p_0 and that $\ulcorner\eta\urcorner$ is set to $\ulcorner\eta_0\urcorner$ during iteration s_0 of the main loop. Also, no number witnesses $\neg\eta_0$ because η_0 is true and hence $\mathrm{true}(\ulcorner\eta_0\urcorner)$ is an axiom of T^+. Therefore, the main loop enters the 'if' case in all iterations past s_0. In particular, the main loop executes $A_{C_0}^{\ulcorner\eta_0\urcorner}(s)$ in iteration s. Thus $A_{C_0}^{\ulcorner\eta_0\urcorner}(s)\downarrow$ because $\Psi(s)\downarrow$. Conversely, suppose that $(\forall s > s_0)(A_{C_0}^{\ulcorner\eta_0\urcorner}(s)\downarrow)$. By Claim 1, $(\forall s \le s_0)(\Psi(s)\downarrow)$, which is (equivalent to) a true Σ_1 sentence. Thus $T \vdash (\forall s \le s_0)(\Psi(s)\downarrow)$ by Σ_1-completeness. We prove by Σ_1-induction on s that $(\forall s \ge s_0)(\Psi(s)\downarrow)$. We already know that $\Psi(s_0)\downarrow$, which gives the base case. Now assume that $\Psi(s)\downarrow$, and consider the execution of $\Psi(s+1)$. The execution of $\Psi(s+1)$ reaches iteration $s+1$ of the main loop because $\Psi(s)\downarrow$. As argued above, the main loop executes $A_{C_0}^{\ulcorner\eta_0\urcorner}(s+1)$ in iteration $s+1$ because $s+1 > s_0$ and $\mathrm{true}(\ulcorner\eta_0\urcorner)$ is an axiom of T^+. By assumption $A_{C_0}^{\ulcorner\eta_0\urcorner}(s+1)\downarrow$, so $\Psi(s+1)\downarrow$.

Now, from

$$\begin{aligned}
T^+ &\vdash (\forall s > s_0)(\Psi(s)\downarrow) \leftrightarrow (\forall s > s_0)(A_{C_0}^{\ulcorner\eta_0\urcorner}(s)\downarrow),\\
T &\vdash (\forall s \le s_0)(\Psi(s)\downarrow), \text{ and}\\
T &\vdash (\forall s > s_0)(A_{C_0}^{\ulcorner\eta_0\urcorner}(s)\downarrow) \leftrightarrow \mathrm{tot}(A_{C_0}^{\ulcorner\eta_0\urcorner})
\end{aligned}$$

(the last of which is easy to see), we conclude that $T^+ \vdash \mathrm{tot}(\Psi) \leftrightarrow \mathrm{tot}(A_{C_0}^{\ulcorner\eta_0\urcorner})$. Therefore, by Lemma 5.1 item (2),

$$T^+ \vdash \mathrm{tot}(\Psi) \leftrightarrow \bigvee_{\substack{\langle\ulcorner\pi\urcorner, e\rangle \in C_0\\ \pi \text{ is true}}} \mathrm{tot}(\Phi_e).$$

If the disjunction is empty, then $T^+ \vdash \neg\,\mathrm{tot}(\Psi)$. Combining this with the assumption $T^+ \vdash \mathrm{tot}(\Psi)\vee\mathrm{tot}(\Gamma)$ yields $T^+ \vdash \mathrm{tot}(\Gamma)$, which contradicts that $\deg_{T^+}(\Gamma) = \mathbf{b} >_{T^+} \mathbf{0}$. If the disjunction is not empty, then consider each $\langle \ulcorner\pi\urcorner, e\rangle \in C_0$ where π is true. For $\langle \ulcorner\pi\urcorner, e\rangle$ to have been added to C_0, it must be that $T+\pi \vdash (\mathrm{tot}(\Phi_e)\wedge\mathrm{tot}(\Psi)) \to \mathrm{tot}(\Gamma)$. Therefore $T^+ \vdash (\mathrm{tot}(\Phi_e)\wedge\mathrm{tot}(\Psi)) \to \mathrm{tot}(\Gamma)$ because π is true. Thus

$$T^+ \vdash \mathrm{tot}(\Psi) \to \bigvee_{\substack{\langle \ulcorner\pi\urcorner, e\rangle \in C_0 \\ \pi \text{ is true}}} (\mathrm{tot}(\Phi_e) \wedge \mathrm{tot}(\Psi)), \text{ and}$$

$$T^+ \vdash \left(\bigvee_{\substack{\langle \ulcorner\pi\urcorner, e\rangle \in C_0 \\ \pi \text{ is true}}} (\mathrm{tot}(\Phi_e) \wedge \mathrm{tot}(\Psi)) \right) \to \mathrm{tot}(\Gamma).$$

It follows that $T^+ \vdash \mathrm{tot}(\Psi) \to \mathrm{tot}(\Gamma)$. Combining this with the assumption $T^+ \vdash \mathrm{tot}(\Psi) \vee \mathrm{tot}(\Gamma)$ yields $T^+ \vdash \mathrm{tot}(\Gamma)$, which again contradicts that $\deg_{T^+}(\Gamma) = \mathbf{b} >_{T^+} \mathbf{0}$. Thus $T^+ \nvdash \mathrm{tot}(\Psi) \vee \mathrm{tot}(\Gamma)$, as desired. ☐

Claim 3. *If e is such that $T^+ \vdash (\mathrm{tot}(\Phi_e) \wedge \mathrm{tot}(\Psi)) \to \mathrm{tot}(\Gamma)$, then $T^+ \vdash \mathrm{tot}(\Phi_e) \to \mathrm{tot}(\Psi)$.*

Proof of claim. We start by showing that p increases infinitely often in the sense that for every p_0 there is an m such that p is set to p_0 in iteration m of the main loop. First, Ψ is total by Claim 1, so Ψ never diverges during the execution of the main loop. Second, p increases exactly in iterations where the main loop enters the 'else' case. Thus if p increases only finitely often, there must be an m_0 such that main loop only enters the 'if' case in iterations past m_0. For this to happen, there must be a true Π_1 sentence η' such that $T+\eta' \vdash \mathrm{tot}(\Psi) \vee \mathrm{tot}(\Gamma)$. Thus $T^+ \vdash \mathrm{tot}(\Psi) \vee \mathrm{tot}(\Gamma)$, which contradicts Claim 2.

Now, suppose that $T^+ \vdash (\mathrm{tot}(\Phi_e) \wedge \mathrm{tot}(\Psi)) \to \mathrm{tot}(\Gamma)$, and let p_0 be a proof witnessing that $T+\pi \vdash (\mathrm{tot}(\Phi_e) \wedge \mathrm{tot}(\Psi)) \to \mathrm{tot}(\Gamma)$ for some true Π_1 sentence π. Let s_0 be such that p is increased from p_0 to p_0+1 during iteration s_0 of the main loop, so that $\langle \ulcorner\pi\urcorner, e\rangle$ is added to C during this iteration. We now argue in T^+ that $\mathrm{tot}(\Phi_e) \to \mathrm{tot}(\Psi)$. As argued in Claim 2, $(\forall s \leq s_0)(\Psi(s)\downarrow)$ is (equivalent to) a true Σ_1 sentence, so $T \vdash (\forall s \leq s_0)(\Psi(s)\downarrow)$ by Σ_1-completeness. We prove by Σ_1-induction on s that $(\forall s \geq s_0)(\Psi(s)\downarrow)$. We already know that $\Psi(s_0)\downarrow$, which gives the base case. Now assume that $\Psi(s)\downarrow$, and consider the execution of $\Psi(s+1)$. The execution of $\Psi(s+1)$ reaches iteration $s+1$ of the main loop because $\Psi(s)\downarrow$. If iteration $s+1$ enters the 'else' case, then clearly $\Psi(s+1)\downarrow$. If iteration $s+1$ enters the 'if' case, then it executes $A_C^{\ulcorner\eta\urcorner}(s+1)$. However, $\langle \ulcorner\pi\urcorner, e\rangle \in C$ because it entered C during iteration $s_0 < s+1$. Thus from the true Π_1 sentence $\mathrm{true}(\ulcorner\pi\urcorner)$ (which is an axiom of T^+), the assumption $\mathrm{tot}(\Phi_e)$, and Lemma 5.1 item (i), we conclude that $A_C^{\ulcorner\eta\urcorner}(s+1)\downarrow$. Thus $\Psi(s+1)\downarrow$. This completes the induction. Finally, we conclude $\mathrm{tot}(\Psi)$ from $(\forall s \leq s_0)(\Psi(s)\downarrow)$ and $(\forall s \geq s_0)(\Psi(s)\downarrow)$. Thus $T^+ \vdash \mathrm{tot}(\Phi_e) \to \mathrm{tot}(\Psi)$, as desired. ☐

Let $\mathbf{a} = \deg_{T^+}(\Psi)$. Then $\mathbf{a} \cap \mathbf{b} >_{T^+} \mathbf{0}$ by Claim 2. If $\mathbf{c}$ is such that $\mathbf{a} \cup \mathbf{c} \geq_{T^+} \mathbf{b}$, then $\mathbf{c} \geq_{T^+} \mathbf{a}$ by Claim 3, so $\mathbf{c} \geq_{T^+} \mathbf{b}$. ☐

Corollary 5.3. *Let T be a recursively axiomatizable extension of $\mathrm{I}\Sigma_1$. Then every $\mathbf{b} \in \mathcal{P}_{T^+}$ with $\mathbf{b} >_{T^+} \mathbf{0}$ has the anti-cupping property.*

Proof. Given $\mathbf{b} >_{T^+} \mathbf{0}$, by Theorem 5.2, let $\mathbf{x}$ be such that $\mathbf{x} \cap \mathbf{b} >_{T^+} \mathbf{0}$ and $(\forall \mathbf{c})[(\mathbf{x} \cup \mathbf{c} \geq_{T^+} \mathbf{b}) \to (\mathbf{c} \geq_{T^+} \mathbf{b})]$. Let $\mathbf{a} = \mathbf{x} \cap \mathbf{b}$. Then $\mathbf{0} <_{T^+} \mathbf{a} <_{T^+} \mathbf{b}$. Now consider a $\mathbf{c} \in \mathcal{P}_{T^+}$ such that $\mathbf{a} \cup \mathbf{c} = \mathbf{b}$. Clearly $\mathbf{c} \leq_{T^+} \mathbf{b}$. On the other hand, using the fact that $\mathcal{P}_{T^+}$ is a distributive lattice,

$$\mathbf{b} = \mathbf{a} \cup \mathbf{c} = (\mathbf{x} \cap \mathbf{b}) \cup \mathbf{c} = (\mathbf{x} \cup \mathbf{c}) \cap (\mathbf{b} \cup \mathbf{c}) = (\mathbf{x} \cup \mathbf{c}) \cap \mathbf{b}.$$

Thus $\mathbf{x} \cup \mathbf{c} \geq_{T^+} \mathbf{b}$, which implies that $\mathbf{c} \geq_{T^+} \mathbf{b}$ by the choice of $\mathbf{x}$. Thus $\mathbf{c} = \mathbf{b}$. This shows that $\mathbf{a}$ does not cup to $\mathbf{b}$, so $\mathbf{a}$ witnesses that $\mathbf{b}$ has the anti-cupping property. ☐

Theorem 5.2 and Corollary 5.3 also hold with T in place of T^+. In this situation, the definition of Ψ can be simplified because there is no longer any need for the π's and η's. One must be careful to check that a similar verification can be done using only T.

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