OPERATOR COVARIANT TRANSFORM AND LOCAL PRINCIPLE

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ABSTRACT. We describe connections between the localization technique introduced by I.B. Simonenko and operator covariant transform produced by nilpotent Lie groups.

CONTENTS

1. INTRODUCTION

In 1965 I.B. Simonenko pioneered [22, 23] localization technique in the theory of operators. It still remains an important tool in this area, see for example [2, 4, 8, 9, 21, 24]. Many questions addressed by this technique, e.g. boundary value problems, are rooted in mathematical physics. We also discuss connections with quantum mechanics in the closing section of this paper.

The localisation method was developed in various directions and there is no possibility to mention all works based on numerous existing variants and modifications of the localization technique. Several generalizations, e.g. within C^*-algebras setup [3, Prop. 4.5], capture the abstract skeleton of the localization technique. However the idea of "localization" has an explicit geometrical meaning, which often escapes those general schemes.

We present here a different point of view on the original works of Simonenko, which highlights the rôle of groups in the constructions. Thus it is not a generalization but rather an attempt to link certain geometrical meaning of locality with homogeneous structure of nilpotent Lie groups. This paper grown up from our earlier works [10–16] revised in the light of recent research [17, 18].

The paper outline is as follows: Section 2 collects preliminary information from other works, which will be used here. In Section 3 we use homogeneous structure of nilpotent Lie groups to define basic elements of localization. Operators which

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are invariant under certain group action are main building blocks for localization, we demonstrate this in Section 4. The final Section 5 offers summary of our observations which lead to new directions for further research.

2. Preliminaries

2.1. **Classic Localization Technique.** We present here the fundamental definitions from the work of I.B. Simonenko [22,23] formulated for operators on $L_p(\mathbb{R}^n)$. Essential norm of an operator is defined by

$$|||A||| = \inf_K \|A - K\|,$$

where the infimum is taken over all compact operators K. For a measurable set $F \subset \mathbb{R}^n$ we define the projection operator $P_F : L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n)$ by:

$$[P_F f](x) = \begin{cases} f(x), & \text{if } x \in F; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

The operators, most suitable for the localization method, are defined as follows.

Definition 1. [22, § I.1] An operator A is of *local type* if for any two closed disjoint sets F_1 and F_2 the operator $P_{F_1} A P_{F_2}$ is compact.

The cornerstone definition for the whole theory is

Definition 2. [22, § I.2] Operators $A, B : L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n)$ are called *equivalent* at a point x_0 if for any $\varepsilon > 0$ there is a neighborhood u of x_0 such that $|||AP_u - BP_u||| < \varepsilon$ and $|||P_u A - P_u B||| < \varepsilon$. This is denoted $A \overset{x_0}{\sim} B$.

As usual there are two stages in this method: analysis and synthesis. Local equivalence decomposes operators into families of local representatives. Now we define the opposite process of a reconstruction.

Definition 3. [22, § I.5] Let A_x be a family of operators $L_p(X) \to L_p(X)$ depending from $x \in X$. An operator $A : L_p(X) \to L_p(X)$ is an *envelope* of A_x if for every x we have $A \overset{x}{\sim} A_x$.

An envelope can be build [22, § I.5] as the limit A of a sequence A_n which is defined by the expression:

$$A_n = \sum_{j=1}^{n} P_{u_j} A_{x_j} P_{u_j}, \tag{2}$$

where sets u_n make a decomposition of X and $x_n \in u_n$.

2.2. **Covariant Transform.** The following concept is a natural development of the coherent states (wavelets) based on group representations.

Definition 4. [17,18] Let ρ be a representation of a group G in a space V and F be an operator from V to a space U. We define a *covariant transform* $\mathcal{W}$ from V to the space $L(G, U)$ of U-valued functions on G by the formula:

$$\mathcal{W} : v \mapsto \hat{v}(g) = F(\rho(g^{-1})v), \qquad v \in V,\ g \in G. \tag{3}$$

Operator F will be called *fiducial operator* in this context.

We borrow the name for operator F from fiducial vectors of Klauder and Skagerstam [20]. The wavelet transform, which is a particular case of the covariant transform, corresponds to the fiducial operator which is a linear functional. Thus its image consists scalar-valued functions. It seems to be most favorable situation, cf. [18, Rem. 3], and was believed to be the only possible one for a long time. A moral of the present work is that the covariant transform can be useful even in the other extreme limit: if the range of the fiducial operator is the entire space V.

By the way, we do not require that the fiducial operator F shall be linear in general, however it will be always linear in the present work. Sometimes the positive homogeneity, i.e. $F(tv) = tF(v)$ for $t > 0$, alone can be already sufficient, see [18, 19].

The following property is inherited by the coherent transform from the wavelet one.

Theorem 5. [17, 18] *The covariant transform* (3) *intertwines* ρ *and the left regular representation* Λ *on* $L(G, U)$*:*

$$\mathcal{W}\rho(g) = \Lambda(g)\mathcal{W}.$$

Here Λ *is defined as usual by:*

$$\Lambda(g) : f(h) \mapsto f(g^{-1}h). \tag{4}$$

The next result follows immediately.

Corollary 6. *The image space* $\mathcal{W}(V)$ *is invariant under the left shifts on* G.

2.3. **Inverse Covariant Transform.** An object invariant under the left action Λ (4) is called *left invariant*. For example, let L and L' be two left invariant spaces of functions on G. We say that a pairing $\langle\cdot,\cdot\rangle : L \times L' \to \mathbb{C}$ is *left invariant* if

$$\langle \Lambda(g)f, \Lambda(g)f' \rangle = \langle f, f' \rangle, \quad \text{for all} \quad f \in L,\ f' \in L'. \tag{5}$$

Remark 7.
(1) We do not require the pairing to be linear in general.
(2) If the pairing is invariant on space $L \times L'$ it is not necessarily invariant (or even defined) on the whole $C(G) \times C(G)$.
(3) An invariant pairing on G can be obtained from an invariant functional l by the formula $\langle f_1, f_2 \rangle = l(f_1 \bar{f}_2)$. Such a functional are often associated to the (quasi-) invariant measures.

Example 8. Let G be the $ax + b$ group, cf. Ex. 12 below. There are essentially two non-trivial invariant pairings for it. The first one is based on the left Haar measure $\frac{da\,db}{a^2}$ and integration over the entire group:

$$\langle f_1, f_2 \rangle = \int_{-\infty}^{\infty}\int_{0}^{\infty} f_1(a,b)\,\bar{f}_2(a,b)\,\frac{da\,db}{a^2}. \tag{6}$$

Another invariant pairing on G, which is not generated by the Haar measure, is:

$$\langle f_1, f_2 \rangle = \lim_{a\to 0}\int_{-\infty}^{\infty} f_1(a,b)\,\bar{f}_2(a,b)\,db. \tag{7}$$

This pairing participates in the definition of the inner product on the Hardy space, thus we call it *Hardy-type pairing* [18].

For a representation ρ of G in V and $v_0 \in V$ we fix a function $w(g) = \rho(g)v_0$. We assume that the pairing can be extended in its second component to this V-valued functions, say, in the weak sense.

Definition 9. Let $\langle\cdot,\cdot\rangle$ be a left invariant pairing on $L \times L'$ as above, let ρ be a representation of G in a space V, we define the function $w(g) = \rho(g)v_0$ for $v_0 \in V$. The *inverse covariant transform* $\mathcal{M}$ is a map $L \to V$ defined by the pairing:

$$\mathcal{M} : f \mapsto \langle f, w\rangle, \qquad \text{where } f \in L. \tag{8}$$

There is an easy consequence of this definition.

Proposition 10. *The inverse wavelet transform intertwines the left regular representation and* $\rho(g)$.

3. Semidirect Products and Localization

Let G be an m-dimensional exponential nilpotent Lie group of the length k. That means that

- we can identify G with its Lie algebra $\mathfrak{g} \sim \mathbb{R}^m$ through the exponential map;
- there is a linear space decomposition

$$\mathfrak{g} = \oplus_{j=1}^{k} V_j, \qquad \text{such that} \quad [V_i, V_j] \in V_{i+j}, \tag{9}$$

  where $[V_i, V_j]$ denotes the space of all commutators $[x, y] = xy - yx$ with $x \in V_i$, $y \in V_j$ and $V_l = \{0\}$ for all $l > k$.

Example 11. Here are two most fundamental examples.

(1) The group of Euclidean shifts in $\mathbb{R}^n$—a nilpotent group of the length 1.
(2) The Heisenberg group $\mathbb{H}^n$ [6,7]—a nilpotent group of dimensionality $m = 2n + 1$ and the length 2. Its element is (s, x, y), where $x, y \in \mathbb{R}^n$ and $s \in \mathbb{R}$. The group law on $\mathbb{H}^n$ is given as follows:

$$(s, x, y) \cdot (s', x', y') = (s + s' + \tfrac{1}{2}(xy' - x'y), x + x', y + y'). \tag{10}$$

For a generic group G described above there is a one-parameter group of automorphisms of $\mathfrak{g}$ defined in terms of decomposition (9):

$$\tau_t(v_j) = t^j v_j, \qquad \text{for} \quad v_j \in V_j,\ t \in \mathbb{R}_+.$$

The exponential map sends τ_t to automorphisms of the group G by the Baker–Campbell–Hausdorff formula. Thus we consider the semidirect product $\bar{G} = G \rtimes \mathbb{R}_+$ of the group G and positive reals with the group law:

$$(t, g) \cdot (t', g') = (tt', g \cdot \tau_t(g')), \qquad \text{where } t, t' \in \mathbb{R}_+,\ g, g' \in G.$$

The unit in $\bar{G}$ is $(1, e)$ and $(t, g)^{-1} = (t^{-1}, \tau_{t^{-1}}(g^{-1}))$.

Example 12. Returning to groups introduced in Example 11:

(1) If G is the group of shifts on the real line $\mathbb{R}$ then the above semidirect $\bar{G}$ product is the $ax + b$ group (or *affine* group). The group $\bar{G}$ is isomorphic to $\mathbb{R}_+ \times \mathbb{R}$ with the group law:

$$(a, b) \cdot (a', b') = (aa', ab' + b), \qquad \text{where } a, a' \in \mathbb{R}_+,\ b, b' \in \mathbb{R}.$$

(2) For the Heisenberg group $\mathbb{H}^n$ the above automorphisms is $\tau_t(s,x,y) = (t^2 s, tx, ty)$ [5], thus the respective group law on $\bar{\mathbb{H}}^n$ is:

$$(t,s,x,y)\cdot(t',s',x',y') = (tt', s + t^2 s' + \frac{t}{2}(xy' - x'y), x + tx', y + ty'). \tag{11}$$

There is a linear action of $\bar{G}$ on functions over G cooked by the "$ax+b$-recipe":

$$[\rho(t,g)f](g') = t^{\frac{k}{p}}\, f(\tau_{t^{-1}}(g^{-1}\cdot g')), \tag{12}$$

where $k = \sum_j j \cdot \dim V_j$. This action is an isometry of $L_p(G) = L_p(G, d\mu)$, where $d\mu$ is the Haar measure on G (recall that it is unimodular as a nilpotent one). Then we can define the respective representation ρ_d of $\bar{G}\times\bar{G}$ on operators [16, 18, 19]:

$$\rho_d(t,g;t',g') : A \mapsto \rho(t^{-1}, \tau_{t^{-1}}(g^{-1}))A\rho(t',g'), \tag{13}$$

for a linear operator $A : L_p(G) \to L_p(G)$.

Let $F_e \subset G$ be a bounded closed subset, which contains a neighbourhood of the unit $e \in G$. We will denote by $F_{(t,g)} = (t,g)\cdot F_e$ for $(t,g) \in \bar{G}$, its image under the left action of $\bar{G}$ on G. Define the associated projection $P_e = P_{F_e}$ by (1). It is a straightforward verification that

$$\rho_d(t,g;t,g)P_e = P_{F_{(t,g)}}, \qquad \text{where } F_{(t,g)} = (t,g)\cdot F_e. \tag{14}$$

We shall use a simpler notation $P_{(t,g)} = P_{F_{(t,g)}}$ again. The exact form of F_e is not crucial for the following construction, but the following property simplifies technical issues:

Definition 13. We say that F_e is r-*self-covering* if for any two intersecting sets $F_{(1,g_1)}$ and $F_{(1,g_2)}$ there is such $g \in G$ that $F_{(r,g)}$ covers the union of $F_{(1,g_1)}$ and $F_{(1,g_2)}$.

For example, the closed unit ball in $\mathbb{R}^n$ is 2-self-covering with no other F_e having a smaller value of r for the self-covering property.

For a Banach space V, we denote by $B(V)$ the collection of all bounded linear operators $V \to V$.

Definition 14. We select a fiducial operator $F : B(L_p(G)) \to B(L_p(G))$ by the identity

$$F(A) = P_e A P_e, \qquad \text{where } A \in B(L_p(G)). \tag{15}$$

Then *Simonenko presymbol* $\hat{S}_A(t,g;t',g')$ of an operator A is the covariant transform (3) generated by the representation ρ_d (13) and the fiducial operator F (15):

$$\begin{aligned}\hat{S}_A(t,g;t',g') &= F(\rho_d(t,g;t',g')A)\\ &= P_e\,\rho(t^{-1},\tau_{t^{-1}}(g^{-1}))\,A\,\rho(t',g')\,P_e.\end{aligned}$$

Thus the Simonenko presymbol is $B(L_p(G))$-valued function on $\bar{G}\times\bar{G}$. We can consider a definition of the alternative presymbol:

$$\tilde{S}_A(t,g;t',g') = P_{(t,g)}\,A\,P_{(t',g')}, \tag{16}$$

which is closer to the original geometrical spirit of Simonenko's works [22, 23]. However there is an easy explicit connection between them:

$$\hat{S}_A((t,g)^{-1};(t',g')^{-1}) = \rho_d(t,g;t',g')\,\tilde{S}_A\,(t,g;t',g'),$$

which is a local transformation of the function value at every point. Thus both symbols shall bring equivalent theories, although each of them seems to be more suitable for particular purposes.

For operators of local type the whole presymbol is excessive due to the following result.

Proposition 15. *Let* F_e *be* r*-self-similar and* A *be an operator of local type. Then for any reals* $t > t' > 0$ *and* $g \in G$ *the operator* $\hat{S}_A(t_1, g_1; t_2, g_2)$ *with* $t_i > t$, $i = 1, 2$ *can be expressed as a finite sum*

$$\hat{S}_A(t_1, g_1; t_2, g_2) = \sum_{k=1}^{n} B_k \hat{S}_A(t', h_k; t', h_k) C_k, \tag{17}$$

for some $h_k \in F_{(t_1,g_1)} \cup F_{(t_2,g_2)}$ *and constant operator coefficients* B_k *and* C_k*, which do not depend on* A.

Proof. We will proceed in terms of the equivalent presymbol $\tilde{S}_A$ (16) since it better reflects geometrical aspects. We also note that if we obtain the decomposition

$$\tilde{S}_A(t_1, g_1; t_2, g_2) = \sum_{k=1}^{n} B_k \tilde{S}_A(t_k, h_k; t_k, h_k) C_k,$$

with all $t_k \leqslant t'$ then we will be able to replace t_k by t' with the simultaneous change of coefficient B_k and C_k in order to get the required identity (17).

Now we put $t'' = t'/r$ and find a finite covering of the compact sets $F_{(t_1,g_1)} \cup F_{(t_2,g_2)}$ by the interiors of sets $F_{(t'',h_k)}$ with $h_k \in F_{(t_1,g_1)} \cup F_{(t_2,g_2)}$. Using the inclusion-exclusion principle we can write:

$$\begin{aligned} P_{(t_i,g_i)} &= \sum_k P_{(t'',h_k)} - \sum_{k,l} P_{(t'',h_k)} P_{(t'',h_l)} + \ldots \\ &\quad - \sum_k P_{(t'',h_k)} P^{\perp}_{(t_i,g_i)} + \sum_{k,l} P_{(t'',h_k)} P_{(t'',h_l)} P^{\perp}_{(t_i,g_i)} - \ldots, \end{aligned}$$

where all sums are finite and the number of sums is finite as well. Moreover each term in the summation contains at least one projection $P_{(t'',h_k)}$. We use this decomposition for the presymbol $P_{(t_1,g_1)} A P_{(t_2,g_2)}$ of an operator A of local type. Then we need to take care only on the terms $P_{(t'',h_k)} A P_{(t'',h_l)}$ where $F_{(t'',h_k)}$ and $F_{(t'',h_n)}$ intersect. Due to the r-self-covering property each such term can be represented as $B_m P_{(t',h_m)} A P_{(t',h_m)} C_m$ for some $h_m \in F_{(t_1,g_1)} \cup F_{(t_2,g_2)}$ with B_m and C_m depending on the geometry of sets only. □

Thus for the operators of local type we give the following definition.

Definition 16. For an operator A of local type we define *Simonenko symbol* $S_A(t, g) = \hat{S}_A(t, g; t, g)$, that is:

$$S_A(t, g) = P_e\, \rho(t^{-1}, \tau_{t^{-1}}(g^{-1}))\, A\, \rho(t, g)\, P_e.$$

Corollary 17. *For an operator* A *of local type the value of the presymbol* $\hat{S}_A(t', g'; t'', g'')$ *at a point* $(t', g'; t'', g'') \in \bar{G} \times \bar{G}$ *is completely determined by the values of symbol* $S_A(t, g)$*,* $g \in G$ *for an arbitrary fixed* t *such that* $t \leqslant \min(t', t'')$.

Corollary 18. *Tho operators* A *and* B *of local type are equal if and only if for any* $\varepsilon > 0$ *there is a positive* $t < \varepsilon$ *such that* $S_A(t, g) = S_B(t, g)$ *for all* $g \in G$.

In other words even the symbol $S_A(t,g)$ contains an excessive information: in a sense we shall look for values of $\lim_{t\to 0} S_A(t,g)$ only. We conclude this section by the restatement of the Definition 2.

Definition 19. Two operators A and B of local type are *equivalent* at a point $g \in G$, denoted by $A \overset{g}{\sim} B$, if

$$\lim_{t\to 0} |||S_{A-B}(t,g)||| = 0.$$

4. Localization and Invariance

The paper of Simonenko [23] already contains results which can be easily adopted to covariant transform setup. This was already used in our previous work [10–15] to study singular integral operators on the Heisenberg group. In this section we provide such restatements of results in term of the representation from (12). Proofs will be omitted since they are easy modifications of the original ones [23].

Definition 20. An operator is called *homogeneous* if it commutes with all transformations $\rho(t,e)$, $t \in \mathbb{R}_+$ (12). If an operator commutes with $\rho(1,g)$, $g \in G$ (12) then it is called *shift-invariant*.

There is an immediate consequence of Thm. 5.

Corollary 21. *The symbols of a homogeneous (or shift-invariant) operator is a function on* $\bar{G}$, *which is invariant under the action of the subgroup* $\mathbb{R}_+ \subset \bar{G}$ *(or* $G \subset \bar{G}$ *respectively).*

Thus homogeneous shift-invariant operators have constant symbols. Tame behavior of operators from those classes is described by the following statements, cf. [23, § II.2].

Lemma 22. *For two homogeneous operators* A *and* B *the following are equivalent:*

(1) $A \overset{e}{\sim} B$, *where* $e \in G$ *is the unit;*
(2) $S_A(t,e) = S_B(t,e)$ *for certain* $t \in \mathbb{R}_+$;
(3) $A = B$.

Lemma 23. [23, § II.2] *For two homogeneous shift-invariant operators* A *and* B *the following are equivalent:*

(1) $A \overset{g}{\sim} B$ *for certain* $g \in G$;
(2) $S_A(t,g) = S_B(t,g)$ *for certain* $(t,g) \in \bar{G}$;
(3) $A = B$.

A shift-invariant operator on G can be associated to a convolution. A convolution, which is also a homogeneous operator, shall have singular kernels. A study of such convolutions can be carried out by means of (non-commutative) harmonic analysis on G. For the (commutative) Euclidean group this was illustrated in [22, 23]. A non-commutative example of the Heisenberg group can be found in [10–15]. It is also possible to study this operators through further versions of wavelets (coherent) transform, e.g. the Berezin-type symbols [16]. In the common case boundedness of the Berezin symbols corresponds to the boundedness of the operator, and if the symbol vanishes at the infinity then the operator is compact.

Once a good description of singular convolutions is obtained (through covariant transform or several such transforms applied in a sequence) we can consider the class of operators which can be reduced to them.

Definition 24. [23, § III.1] A linear operator A of local type is called a *generalized singular integral* if A is equivalent at every point of G to a some homogeneous shift-invariant operator.

The final step of the construction is synthesis of an operator from the field of local representatives using the inverse covariant transform from Subsection 2.3. To this end we need to chose an invariant pairing on the group $\bar{G}$, keeping the $ax+b$ group as an archetypal example. For operators of local type the whole information is concentrated in the arbitrary small neighborhood of the subgroup $G \subset \bar{G}$, cf. Cor. 18. Thus we select the Hardy-type functional (7) instead of the Haar one (6). Let $d\mu$ be the Haar measure on the group G. Then the following integral

$$\langle f_1, f_2 \rangle = \lim_{t\to 0} \int_G f_1(t,g) f_2(t,g)\, d\mu(g), \tag{18}$$

defines an invariant pairing on the group $\bar{G}$.

We again make use of the fiducial operator $F(A) = P_e A P_e$ (15). In the language of wavelet theory we may say that analyzing and reconstructing vectors are the same. The respective transformation $\rho_f(t,g)F$ by an element of the group $\bar{G}$ is defined through the identity $[\rho_f(t,g)F](A) = P_{(t,g)} A P_{(t,g)}$ for an arbitrary A. Consequently the inverse covariant transform (8) sends an operator valued function $A(t,g)$ to an operator through the invariant pairing:

$$\mathcal{M} : A(t,g) \mapsto A = \lim_{t\to 0} \int_G P_{(t,g)} A(t,g) P_{(t,g)}\, d\mu(g).$$

The last integral may be realized through Riemann-type sums which are lead to the approximation (2) of an envelope of $A(t,g)$.

5. Closing Remarks

In this work we outlined an interpretation of the classical Simonenko's localization method [22, 23] in the context of recently formulated covariant transform [17, 18]. The original localization was used to study singular integral operators, which are convolutions on the Euclidean group. Our interpretation allows to make a straightforward modification of localization technique for non-commutative nilpotent Lie groups. The crucial role is played by the one-parameter group of automorphism realized as dilations.

Once local representatives are obtained they can be studied further by other forms of wavelet (covariant) transform. The Berezin symbol seems to be very suitable for this task. Such a chain (Simonenko–Berezin–. . .) of covariant transforms shall lead to the full dissection of initial operator into a very detailed symbol, which may be even scalar valued. The opposite process, reconstruction of an operator from its symbol or local representatives, can be done by the inverse covariant transform, which uses the same group structure.

The original coherent states in quantum mechanics are obtained from the ground state of the harmonic oscillator by a unitary action of the Weyl–Heisenberg group [1, Ch. 1]. The next standard move is a decomposition of an arbitrary state into a linear superposition of coherent states, which form an overcomplete set. Consequently, observables can be investigated through such decompositions of states.

However, observables are primary notions of quantum theory, thus direct techniques, which circumvent decomposition of states, look more preferable. Classical coherent states have the best possible (within the Heisenberg uncertainty relations) localisation in the phase space. Thus our localisation on nilpotent Lie groups, in particular the Heisenberg group, has a particular significance for quantum theory. Any observable corresponding to an operator of local type can be represented as a compact operator and a continuous field of local representatives. Compact operators have a discrete spectrum with a complete set of eigenvectors each having at most a finite degeneracy. Local representatives corresponds to observables which are highly localised on the phase space. Thus operators of local type is a large set of quantum observables admitting efficient calculations of their spectrum.

It would be interesting to look for a similar construction in other classes of Lie groups. For example, Toeplitz operators on the Bergman space [24] may be treated through the group $\mathrm{SL}_2(\mathbb{R})$ [19], which is semisimple. Such groups do not admit a group of dilation-type global automorphisms, thus some adjustments to the scheme are required at this point.

Another interesting direction of development is operators of non-local type. They may look very different from the view-point of geometrical localization, however it terms of covariant transform the distinction is not so huge. For operators of local type their Simonenko presymbol over $\bar{G} \times \bar{G}$ is excessive and we can consider only the symbol in a small vicinity of the boundary G of the diagonal in $\bar{G} \times \bar{G}$. For operators of non-local type the presymbol on the whole group $\bar{G} \times \bar{G}$ shall be used. This topic deserves a further consideration.

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