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# Location and Orientation Optimisation for Spatially Stretched Tripole Arrays Based on Compressive Sensing

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**Abstract**—The design of sparse spatially stretched tripole arrays is an important but also challenging task and this paper proposes for the very first time efficient solutions to this problem. Unlike for the design of traditional sparse antenna arrays, the developed approaches optimise both the dipole locations and orientations. The novelty of the paper consists in formulating these optimisation problems into a form that can be solved by the proposed compressive sensing and Bayesian compressive sensing based approaches. The performance of the developed approaches is validated and it is shown that accurate approximation of a reference response can be achieved with a 67% reduction in the number of dipoles required as compared to an equivalent uniform spatially stretched tripole array, leading to a significant reduction in the cost associated with the resulting arrays.

**Index Terms**—Sparse array, spatially stretched, tripole, compressive sensing, Bayesian compressive sensing.

## I. INTRODUCTION

### A. Related Work

For uniform linear arrays (ULAs), an adjacent antenna separation of no larger than half of the operating wavelength is used to avoid the introduction of grating lobes [1], [2]. This can become prohibitive in terms of the cost associated with the number of antennas required. Instead, sparse arrays become a desirable alternative due to the fact that the nonuniform nature of their adjacent antenna separations avoids grating lobes even when the mean adjacent antenna separation is greater than half the operating wavelength [3].

However, the sidelobe behaviour of sparse arrays is unpredictable. This means that optimisation of the antenna locations is required in order to achieve a desired beam response. Such optimisation can be achieved by stochastic optimisation methods such as genetic algorithms (GAs) [4]–[6], and simulated annealing (SA) [7], [8]. Difference sets and almost difference sets have also been successfully used in the design of sparse arrays, [9], [10], and merged with GAs to help give an improved performance, [11], [12]. The disadvantage of GAs, and similar design methods, is the potentially long computation time and the possibility of convergence to a non-optimal solution.

More recently, the area of compressive sensing (CS) has been explored [13], and CS-based methods have been proposed in the design of traditional sparse arrays [14]–[19]. CS

theory says that when certain conditions are met it is possible to recover some signals from fewer measurements than used by traditional methods [13]. It is possible to use CS to design sparse sensor arrays by obtaining a close approximation of a desired beam response using as few array elements as possible.

Further work has also shown that it is possible to improve the sparseness of a solution by considering a reweighted  $l_1$  norm minimisation problem [17], [20]–[22]. The aim of these methods is to bring the minimisation of the  $l_1$  norm of the weight coefficients closer to that of the minimisation of the  $l_0$  norm. To do this an iterative method is required to solve a series of reweighted  $l_1$  minimisation problems, where locations with small weight coefficients are more heavily penalised than locations with large weight coefficients.

Alternatively, the problem can be converted into a probabilistic framework (termed Bayesian compressive sensing (BCS)) [23], with some suggested advantages to BCS as compared to traditional CS based implementations. However, an important point of interest is that the problem can be solved by the relevance vector machine (RVM) optimisation framework [24], which is efficient to use as also supported by the comparisons shown in the design examples section of this paper. Additionally, using BCS can remove the need to fine tune the error limits or sparsity associated with the implementations of CS above [25]. Such approaches have been applied in the design of sparse arrays with real valued and complex valued weight coefficients [26]–[28], where the multi-task BCS scheme [29], is applied in the case of complex valued weight coefficients.

The methods discussed above have been implemented assuming the arrays consist of isotropic array elements. As a result, the polarisation of a signal is not taken into account when considering the performance of an array. Instead arrays based on vector sensors, [19], [30], provide a desirable alternative as they allow the measurement of both the horizontal and vertical components of the received waveform. For example, the vector sensors used could be crossed dipoles (two orthogonally orientated dipoles) [19], [31]–[33], or tripoles (three orthogonally orientated dipoles) [34], [35].

When tripoles are used it is possible to measure the full electromagnetic (EM) field at a given point [35]. These arrays have been applied in the area of direction and polarisation

estimation [34]. Due to the close proximity of the three orthogonal dipoles that make up each tripole there can be issues with mutual coupling when implemented in practice. As a result, the concept of spatially stretched tripoles (SST) has been developed and used in the area of direction of arrival (DOA) estimation [35]. An SST is a tripole where the three orthogonal dipoles are spread over a given geometry, leading to reduced mutual coupling effects.

## B. Contributions

In this work for the first time the problem of designing sparse SST arrays (SSSTAs) is addressed. Unlike for the design of traditional sparse arrays there are now two optimisation problems to solve, i.e. finding the optimal locations and orientations for the dipoles. It is proposed to use CS and BCS based design methods that go beyond the state of the art in order to solve these problems.

As a result, it is now necessary to formulate the problem to include the fact that there are three potential dipoles at each point on the sampling grid and the signal model now includes polarisation information (requiring alterations to the CS and BCS formulations). It is possible to avoid co-located dipoles by viewing them as a special case of the minimum adjacent dipole separation not meeting a physical size constraint [17]. However, if the methods in [17] are directly applied in this case, then although there will be a minimum spacing between antenna locations, there can still be multiple dipoles at each location. Therefore it is necessary to consider co-located dipoles as breaking the size constraint. Here, the design of SSSTAs utilising the size constraint is implemented in two ways: i) An iterative minimum distance sampling method (IMDSM) with CS and BCS; ii) an altered iterative reweighted minimisation scheme (AIRMS). When integrating the CS/BCS based method with the IMDSM it is also important to account for the response due to the previously fixed dipoles when deciding what the reference response in the current iteration is.

The remainder of this paper is structured as follows: Section II gives details of the proposed design methods, including the array model being used (II-A), a review of CS and BCS (II-B and II-C) and the proposed IMDSM and reweighted design methods for SSSTAs (II-D and II-E). In Section III design examples are presented to verify the effectiveness of the proposed methods and conclusions are drawn in Section IV.

## II. PROPOSED DESIGN METHODS

### A. Array Model

Figure 1 shows an example of a linear SSSTA.  $M$  possible dipole locations are spread along the  $y$ -axis with an adjacent separation of  $d$ . For each possible dipole location there are three potential orientation directions, one parallel to each axis. Also shown is a signal with its direction of arrival (DOA) defined by the angles  $\theta$  and  $\phi$ , with  $0 \leq \theta \leq \pi/2$  and  $-\pi/2 \leq \phi \leq \pi/2$  [34], [35]. A plane-wave signal model is assumed, i.e. the signal impinges upon the array from the far field.

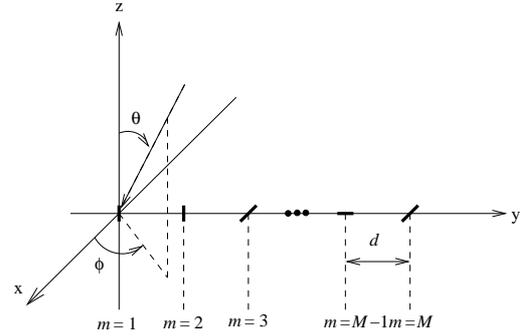


Fig. 1. An example of a spatially stretched tripole array, where the short lines indicate the dipoles and the directions they are orientated in,  $x$ ,  $y$  and  $z$  are the axes of a Cartesian coordinate system and  $M$  is the number of dipoles used.

The spatial steering vector of the array is given by

$$\mathbf{s}_s(\theta, \phi) = [1, \dots, e^{-j2\pi d \sin \theta \sin \phi / \lambda}, \dots, e^{-j2\pi(M-1)d \sin \theta \sin \phi / \lambda}]^T, \quad (1)$$

where  $\lambda$  is the wavelength of interest and  $\{\cdot\}^T$  indicates the transpose operation. The spatial-polarization coherent vector, which contains information about a signal's polarisation and is given by [34], [35]:

$$\begin{aligned} \mathbf{s}_p(\theta, \phi, \gamma, \eta) &= \begin{bmatrix} \sin \gamma \cos \theta \cos \phi e^{j\eta} - \cos \gamma \sin \phi \\ \sin \gamma \cos \theta \sin \phi e^{j\eta} - \cos \gamma \cos \phi \\ -\sin \gamma \sin \theta e^{j\eta} \end{bmatrix} \\ &= \begin{bmatrix} s_{p,x}(\theta, \phi, \gamma, \eta) \\ s_{p,y}(\theta, \phi, \gamma, \eta) \\ s_{p,z}(\theta, \phi, \gamma, \eta) \end{bmatrix} \end{aligned} \quad (2)$$

where  $\gamma \in [0, \pi/2]$  is the auxiliary polarization angle and  $\eta \in [-\pi, \pi)$  is the polarization phase difference.

Now the array can be split into three sub-arrays, one parallel to each axis. With  $f \in \{x, y, z\}$ , the steering vector of each sub-array is given by:

$$\mathbf{s}_f(\theta, \phi, \gamma, \eta) = s_{p,f}(\theta, \phi, \gamma, \eta) \mathbf{s}_s(\theta, \phi). \quad (3)$$

The response of the array is given by

$$p(\theta, \phi, \gamma, \eta) = \mathbf{s}(\theta, \phi, \gamma, \eta)^T \mathbf{w}, \quad (4)$$

with

$$\mathbf{w} = [w_{x,1}, w_{y,1}, w_{z,1}, \dots, w_{x,M}, w_{y,M}, w_{z,M}]^H, \quad (5)$$

where  $w_1 = w_{x,1}$  is the complex weight coefficient for the dipole located at the point  $m = 1$  and orientated parallel to the  $x$ -axis and  $\{\cdot\}^H$  denotes the Hermitian transpose. Note that for an SSSTA if  $w_{x,1} \neq 0$ , then  $w_{y,1} = w_{z,1} = 0$ , as there can be only one dipole present. Similarly

$$\mathbf{s}(\theta, \phi, \gamma, \eta) = [s_{x,1}(\theta, \phi, \gamma, \eta), s_{y,1}(\theta, \phi, \gamma, \eta), s_{z,1}(\theta, \phi, \gamma, \eta), \dots, s_{x,M}(\theta, \phi, \gamma, \eta), s_{y,M}(\theta, \phi, \gamma, \eta), s_{z,M}(\theta, \phi, \gamma, \eta)]^T, \quad (6)$$

where  $s_{x,1}(\theta, \phi, \gamma, \eta)$  is the contribution of the dipole located at the point  $m = 1$  to the overall steering vector parallel to the  $x$ -axis.

### B. Compressive Sensing for SSSTA Design

Suppose  $P_r(\theta, \phi, \gamma, \eta)$  is the desired beam response as a function of  $\theta, \phi, \gamma$  and  $\eta$ . Then the problem is to match the designed response to this desired response for the full range of  $\theta, \phi, \gamma$  and  $\eta$  values of interest while finding the optimised dipole locations and orientations.

First, consider Figure 1 as being a grid of potential dipole locations. Here  $M$  is a large number and sparseness is then introduced by selecting the weight coefficients to give as few active dipoles as possible, or in other words as few non-zero valued weight coefficients as possible, while still giving a designed response close to the desired one. Note, a large  $M$  means it is more likely that the optimal locations will appear on the grid thus allowing for a better performance. However, the tradeoff is that if  $M$  is too large the efficiency of the algorithm deteriorates.

This problem is formulated as

$$\min \|\mathbf{w}\|_1 \quad \text{subject to} \quad \|\mathbf{p}_r - \mathbf{S}\mathbf{w}\|_2 \leq \alpha, \quad (7)$$

where  $\|\mathbf{w}\|_1$  is the  $l_1$  norm of the weight coefficients [13],  $\mathbf{p}_r$  is the vector holding the desired beam response at the sampled angular and polarisation points of interest,  $\mathbf{S}$  is the matrix composed of the corresponding steering vectors, and  $\alpha$  places a limit on the allowed difference between the desired and the designed responses. Minimising the  $l_1$  norm has the effect of minimising the number of dipoles used, while the constraint ensures a reasonable approximation of the ideal reference response is achieved. If the size of  $\alpha$  is increased, more error can be introduced into the final response, which would be expected to allow a sparser solution to be achieved. Note,  $\|\cdot\|_2$  indicates the  $l_2$  norm.

In detail,  $\mathbf{p}_r$  and  $\mathbf{S}$  are respectively given by

$$\mathbf{p}_r = [P_r(\theta_1, \phi_1, \gamma_1, \eta_1), \dots, P_r(\theta_L, \phi_L, \gamma_L, \eta_L)]^T, \quad (8)$$

$$\mathbf{S} = [\mathbf{s}(\theta_1, \phi_1, \gamma_1, \eta_1), \dots, \mathbf{s}(\theta_L, \phi_L, \gamma_L, \eta_L)]^T, \quad (9)$$

where  $L$  is the number of points sampled at each dimension of the desired beam response. In this work  $\mathbf{p}_r$  is the ideal response, i.e. a value of one for the mainlobe and zeros for the other entries. Note,  $L$  has to be large enough to ensure all angular and polarisation points of interest are considered.

Since the coefficients are complex valued, (7) can be reformulated as a modified  $l_1$  norm minimisation [36]:

$$\begin{aligned} \min \quad & q \in \mathbb{R}^+ \\ \text{subject to} \quad & \|\mathbf{p}_r - \mathbf{S}\mathbf{w}\|_2 \leq \alpha, \quad |\langle \mathbf{w} \rangle|_1 \leq q \end{aligned} \quad (10)$$

where

$$|\langle \mathbf{w} \rangle|_1 = \sum_{m=1}^{3M} \|\mathbf{w}_m\|_2 \quad (11)$$

and  $\mathbf{w}_m = [R(w_m), I(w_m)]^T$  for  $m = 1, \dots, 3M$  contains the real and imaginary components of the complex weight coefficient given by the  $m^{\text{th}}$  entry in  $\mathbf{w}$ . Here, the variable  $q$  has been introduced and requires minimising. By keeping  $|\langle \mathbf{w} \rangle|_1$  less than this value the effect is to minimise the  $l_1$  norm of all of the absolute weight coefficients.

Now decompose  $q$  to  $q = \sum_{m=1}^{3M} q_m$ ,  $q_m \in \mathbb{R}^+$ , to reformulate (10). Note, the upper limit on the sum is  $3M$  as there are 3 potential dipole orientations at each location.

In vector form,  $q = \mathbf{1}^T \mathbf{q}$ , where  $\mathbf{1}^T = [1, \dots, 1]$  and  $\mathbf{q} = [q_1, \dots, q_{3M}]^T$ . Then (10) can be rewritten as

$$\begin{aligned} \min_{\mathbf{q}} \quad & \mathbf{1}^T \mathbf{q} \\ \text{subject to} \quad & \|\mathbf{p}_r - \mathbf{S}\mathbf{w}\|_2 \leq \alpha \\ & \|\mathbf{w}_m\|_2 \leq q_m, \quad m = 1, \dots, 3M. \end{aligned} \quad (12)$$

Note, a value of  $q_m = 0$ , means the second constraint in (12) ensures that the real and imaginary parts of the weight coefficient contained in  $\mathbf{w}_m$  will both be equal to zero. This allows the desired sparsity to be introduced.

Now define

$$\hat{\mathbf{w}} = [q_1, R(w_{x,1}), -I(w_{x,1})q_2, \dots, -I(w_{z,M})]^T, \quad (13)$$

$$\hat{\mathbf{c}} = [1, 0, 0, 1, 0, 0, \dots, 1, 0, 0]^T, \quad (13)$$

$$\hat{\mathbf{p}}_r = [R(\mathbf{p}_r), I(\mathbf{p}_r)]^T \quad (14)$$

and

$$\hat{\mathbf{S}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ R(\mathbf{s}_{x,1}) & I(\mathbf{s}_{x,1}) \\ -I(\mathbf{s}_{x,1}) & R(\mathbf{s}_{x,1}) \\ \mathbf{0} & \mathbf{0} \\ R(\mathbf{s}_{y,1}) & I(\mathbf{s}_{y,1}) \\ -I(\mathbf{s}_{y,1}) & R(\mathbf{s}_{y,1}) \\ \vdots & \vdots \\ R(\mathbf{s}_{z,M}) & I(\mathbf{s}_{z,M}) \\ -I(\mathbf{s}_{z,M}) & R(\mathbf{s}_{z,M}) \end{pmatrix}^T, \quad (15)$$

where  $R(\cdot)$  is the real component and  $I(\cdot)$  is the imaginary component. Then, the final formulation is as follows

$$\begin{aligned} \min_{\hat{\mathbf{w}}} \quad & \hat{\mathbf{w}}^T \hat{\mathbf{c}} \\ \text{subject to} \quad & \|\hat{\mathbf{p}}_r - \hat{\mathbf{S}}\hat{\mathbf{w}}\|_2 \leq \alpha \\ & \|\mathbf{w}_m\|_2 \leq q_m, \quad m = 1, \dots, 3M. \end{aligned} \quad (16)$$

Note, the values  $q_m$  for  $m = 1, \dots, 3M$  are included with the weight coefficients in  $\hat{\mathbf{w}}$ . This is so that it is not necessary to predefine their values, instead the algorithm finds them at the same time as the optimised weight coefficients. As a result, it is necessary for the vector  $\hat{\mathbf{c}}$  to select the values  $q_m$  for minimisation and the zeros are introduced into  $\hat{\mathbf{S}}$  to ensure the same values do not contribute to the error between the ideal reference response and the achieved response in the first constraint in (16). Finally, as the weight coefficients have been split into real and imaginary parts, the response given by the product  $\hat{\mathbf{S}}\hat{\mathbf{w}}$  will contain the real and imaginary parts of the achieved response separately. This means the reference pattern has to be split in a similar manner giving (14).

However, unlike the  $l_0$  norm, the  $l_1$  norm does not penalise all non-zero valued coefficients equally. Instead, larger coefficients are penalised more heavily. To further improve the sparseness of the array and get a better approximation of the  $l_0$  norm minimisation, large reweighting terms can be applied to the smaller weight coefficients so that they are penalised more heavily [17], [18], [20]–[22].

When applied to the above modified  $l_1$  norm minimisation problem we get the following

$$\begin{aligned} & \min_{\hat{\mathbf{w}}} \quad \hat{\mathbf{w}}^T \hat{\mathbf{c}} \\ & \text{subject to} \quad \|\hat{\mathbf{p}}_r - \hat{\mathbf{S}}\hat{\mathbf{w}}\|_2 \leq \alpha \\ & \quad \delta_m^i \|\mathbf{w}_m\|_2 \leq q_m, \quad m = 1, \dots, 3M, \end{aligned} \quad (17)$$

where now  $\hat{\mathbf{c}} = [\delta_1^i, 0, 0, \delta_2^i, 0, 0, \dots, \delta_{3M}^i, 0, 0]^T$  and  $\delta_m^i = (|w_m^{i-1}| + \epsilon)^{-1}$ . Here  $i$  is the current iteration,  $\hat{\mathbf{w}}$  holds the current estimate of the weight coefficients,  $w_m^{i-1}$  contains the weight coefficients, from the previous iteration, for the  $m^{\text{th}}$  dipole and  $\epsilon$  is a small value roughly equal to the minimum desired weight coefficient. The iterative algorithm would then follow the steps below:

- 1) Set  $i = 0$  and find an initial estimate of the weight coefficients by solving (16).
- 2)  $i = i + 1$ , and find the reweighting terms  $\delta_m^i$ .
- 3) Solve (17).
- 4) Repeat steps 2 to 3 until  $\|\mathbf{w}^i\|_0 = \|\mathbf{w}^{i-1}\|_0 = \|\mathbf{w}^{i-2}\|_0$  i.e. until the number of active locations has remained the same for three iterations. Here define  $\mathbf{w}^i = [w_1^i, w_2^i, \dots, w_{3M}^i]^T$ .

The addition of the reweighting term, which is calculated using coefficients from the previous iteration, means all non-zero valued coefficients are penalised in a more uniform manner.

It is worth noting that as it stands the solutions to (16) and (17) do not strictly give an SSSTA in the result. This is because currently there is no way of guaranteeing there can only be a single dipole at a given location. In other words the proposed methods are in effect finding a sparse weight coefficient vector without considering the locations of the associated dipoles. The methods detailed in Section II-D and Section II-E can both be used to overcome this issue and ensure that there are no co-located dipoles, guaranteeing an SSSTA.

### C. Bayesian Compressive Sensing for SSSTA Design

When considering BCS for sparse array design, [26]–[28], there are two formulations of BCS that can be used. Firstly there is a single task (ST) BCS formulation [23] which can be implemented using a RVM [24]. Alternatively multi task (MT) BCS, [29], can be used when there are multiple CS measurements and the statistical relationships between them can be exploited. This could include measurements at multiple time instances, or in the case of sparse array design if multiple or complex weight coefficients have to be minimised [27]. As a result MT-BCS is well suited to the problem being addressed and is formulated in what follows.

Firstly, consider matching the real and imaginary parts of the achieved array response to that of the ideal reference response:

$$\hat{\mathbf{p}}_F - \check{\mathbf{S}}\mathbf{w}_F^T = \tilde{\mathbf{D}}_F, \quad (18)$$

where  $F \in \{R, I\}$ ,  $\tilde{\mathbf{D}}_R$  and  $\tilde{\mathbf{D}}_I$  are zero mean Gaussian error vectors, with a variance of  $\check{\sigma}^2$ ,  $\mathbf{w}_R = R(\mathbf{w})$ ,  $\mathbf{w}_I = -I(\mathbf{w})$ ,  $\check{\mathbf{S}} = [R(\mathbf{S})^T, I(\mathbf{S})^T]^T$ ,  $\mathbf{p}_r = \mathbf{p}_R + j\mathbf{p}_I$ ,  $\hat{\mathbf{p}}_R = [R(\mathbf{p}_R), I(\mathbf{p}_R)]^T$  and  $\hat{\mathbf{p}}_I = [R(\mathbf{p}_I), I(\mathbf{p}_I)]^T$ . The problem now is to find the solutions to solve

$$\mathbf{w}_F = \max_{\mathbf{w}_F} \mathcal{P}(\mathbf{w}_F | \hat{\mathbf{p}}_F). \quad (19)$$

It is known that for the likelihood function  $\mathcal{P}(\hat{\mathbf{p}}_F | \mathbf{w}_F)$  and the priors  $\mathcal{P}(\mathbf{w}_F)$  and  $\mathcal{P}(\hat{\mathbf{p}}_F)$ , the following applies

$$\mathcal{P}(\mathbf{w}_F | \hat{\mathbf{p}}_F) = \frac{\mathcal{P}(\hat{\mathbf{p}}_F | \mathbf{w}_F) \mathcal{P}(\mathbf{w}_F)}{\mathcal{P}(\hat{\mathbf{p}}_F)}. \quad (20)$$

This allows the problem to be written as

$$\mathbf{w}_F = \max_{\mathbf{w}_F} \mathcal{P} \left( \frac{\mathcal{P}(\hat{\mathbf{p}}_F | \mathbf{w}_F) \mathcal{P}(\mathbf{w}_F)}{\mathcal{P}(\hat{\mathbf{p}}_F)} \right). \quad (21)$$

The prior  $\mathcal{P}(\mathbf{w}_R)$  is the same as  $\mathcal{P}(\mathbf{w}_I)$  to model the relationship between the real and imaginary parts of the weight coefficients, while still enforcing sparsity. It is given by  $\mathcal{P}(\mathbf{w}_F)$  and found as follows:

$$\mathcal{P}(\mathbf{w}_F) = \int \mathcal{P}(\mathbf{w}_F | \check{\mathbf{a}}, \check{\sigma}^2) \mathcal{P}(\check{\mathbf{a}}) \mathcal{P}(\check{\sigma}^2) d\check{\mathbf{a}} d\check{\sigma}^2, \quad (22)$$

where  $\mathcal{P}(\check{\mathbf{a}})$  is the multi-task shared hyperpriors,  $\check{\mathbf{a}} = [\check{a}_1, \check{a}_1, \dots, \check{a}_1]^T$ , given by a Gamma distribution, and  $\mathcal{P}(\check{\sigma}^2)$  is a shared Gamma hierarchical prior, where

$$\mathcal{P}(\mathbf{w}_F | \check{\mathbf{a}}, \check{\sigma}^2) = (2\pi\check{\sigma})^{-3M} \prod_{m=1}^{3M} \sqrt{\check{a}_m} e^{-\frac{\check{a}_m w_{F,m}^2}{2\check{\sigma}^2}}, \quad (23)$$

gives

$$\mathbf{w}_{F,opt} = \max_{\mathbf{w}_F} \left( \int \frac{\mathcal{P}(\mathbf{w}_F | \check{\mathbf{a}}, \check{\sigma}^2) \mathcal{P}(\hat{\mathbf{p}}_F | \mathbf{w}_F) \mathcal{P}(\check{\mathbf{a}}) \mathcal{P}(\check{\sigma}^2)}{\mathcal{P}(\hat{\mathbf{p}}_F)} d\check{\mathbf{a}} d\check{\sigma}^2 \right), \quad (24)$$

which after integrating over  $\check{\sigma}^2$  and simplifying gives:

$$\mathbf{w}_{F,opt} = \max_{\mathbf{w}_F} \left( \int \mathcal{P}(\mathbf{w}_F | \hat{\mathbf{p}}_F, \check{\mathbf{a}}) \mathcal{P}(\check{\mathbf{a}} | \hat{\mathbf{p}}_F) d\check{\mathbf{a}} \right). \quad (25)$$

Equation (23) considers  $3M$  points as there are three potential dipoles at each location.

Note,

$$\mathcal{P}(\mathbf{w}_F | \hat{\mathbf{p}}_F, \check{\mathbf{a}}) = \int \mathcal{P}(\mathbf{w}_F | \hat{\mathbf{p}}_F, \check{\mathbf{a}}, \check{\sigma}^2) \mathcal{P}(\check{\sigma}^2) d\check{\sigma}^2 \quad (26)$$

and from Bayes' theorem

$$\begin{aligned} & \mathcal{P}(\mathbf{w}_F | \hat{\mathbf{p}}_F, \check{\mathbf{a}}, \check{\sigma}^2) \mathcal{P}(\check{\sigma}^2) = \\ & \frac{\mathcal{P}(\hat{\mathbf{p}}_F | \mathbf{w}_F, \check{\sigma}^2) \mathcal{P}(\mathbf{w}_F | \check{\mathbf{a}}, \check{\sigma}^2) \mathcal{P}(\check{\sigma}^2)}{\int \mathcal{P}(\hat{\mathbf{p}}_F | \mathbf{w}_F, \check{\sigma}^2) \mathcal{P}(\mathbf{w}_F | \check{\mathbf{a}}, \check{\sigma}^2) d\mathbf{w}_F}. \end{aligned} \quad (27)$$

From (23), the fact that a Gamma hierarchical prior is placed on  $\mathcal{P}(\check{\sigma}^2)$  and the fact that  $\mathcal{P}(\mathbf{w}_F | \hat{\mathbf{p}}_F, \check{\sigma}^2)$  can be modelled as a Gaussian likelihood, then

$$\begin{aligned} & \mathcal{P}(\mathbf{w}_F | \hat{\mathbf{p}}_F, \check{\mathbf{a}}) = \left( \int_0^\infty t^{\beta_{MT-1} + (3M/2) - 1} e^{-t} dt \right) \times \\ & \frac{(1 + \frac{1}{2\beta_{MT-2}} (\mathbf{w}_F - \hat{\boldsymbol{\mu}}_F)^T \hat{\boldsymbol{\Sigma}}^{-1} (\mathbf{w}_F - \hat{\boldsymbol{\mu}}_F))^{-(\beta_{MT-2} + (3M/2))}}{\left( \int_0^\infty t^{\beta_{MT-1} - 1} e^{-t} dt \right) (2\pi\beta_{MT-2})^{(3M/2)} \sqrt{\det(\hat{\boldsymbol{\Sigma}})}}, \end{aligned} \quad (28)$$

where  $\beta_{MT-1}$  and  $\beta_{MT-2}$  are parameters associated with the MT-BCS process chosen to encourage sparsity. In (28) the mean and covariance are given by:

$$\hat{\boldsymbol{\mu}}_F = \hat{\boldsymbol{\Sigma}} \check{\mathbf{S}}^T \hat{\mathbf{p}}_F, \quad (29)$$

$$\hat{\boldsymbol{\Sigma}} = \left( \hat{\mathbf{A}} + \check{\mathbf{S}}^T \check{\mathbf{S}} \right)^{-1}, \quad (30)$$

respectively, where  $\hat{\mathbf{A}} = \text{diag}(\check{\mathbf{a}}) = \text{diag}(\check{a}_1, \check{a}_2, \dots, \check{a}_{3M})$ . Note, this gives a Student's t-distribution for  $\mathcal{P}(\mathbf{w}_F | \hat{\mathbf{p}}_F, \check{\mathbf{a}})$ .

When considering the remaining term in (25) a delta function approximation can be used [27]. This is because a closed-form solution is not possible. Note,

$$\begin{aligned} \mathcal{P}(\check{\mathbf{a}} | \hat{\mathbf{p}}_F) &\propto \mathcal{P}(\hat{\mathbf{p}}_F | \check{\mathbf{a}}) \mathcal{P}(\check{\mathbf{a}}) \\ &\propto \left( \int \mathcal{P}(\hat{\mathbf{p}}_F | \mathbf{w}_F, \check{\sigma}^2) \mathcal{P}(\mathbf{w}_F | \check{\mathbf{a}}, \check{\sigma}^2) \right. \\ &\quad \left. \times \mathcal{P}(\check{\sigma}^2) d\mathbf{w}_F d\check{\sigma}^2 \right) \mathcal{P}(\check{\mathbf{a}}), \end{aligned} \quad (31)$$

with a mode given by

$$\check{\mathbf{a}}_{opt} = \max_{\check{\mathbf{a}}} \mathcal{L}(\check{\mathbf{a}}), \quad (32)$$

where

$$\begin{aligned} \mathcal{L}(\check{\mathbf{a}}) = &-\frac{1}{2} \sum_F \left( \log \left( \left| \mathbf{I} + \check{\mathbf{S}} \hat{\mathbf{A}}^{-1} \check{\mathbf{S}}^T \right| \right) + (3M + 2\beta_{MT-1}) \right. \\ &\left. \times \log \left( \hat{\mathbf{p}}_F^T (\mathbf{I} + \check{\mathbf{S}} \hat{\mathbf{A}}^{-1} \check{\mathbf{S}}^T) \hat{\mathbf{p}}_F + 2\beta_{MT-2} \right) \right). \end{aligned} \quad (33)$$

As the mode of a student-t distribution is equal to its mean the resulting weight coefficients are given by [27]

$$\mathbf{w}_{F,opt} = \left( \text{diag}(\check{\mathbf{a}}_{opt}) + \check{\mathbf{S}}^T \check{\mathbf{S}} \right)^{-1} \check{\mathbf{S}}^T \hat{\mathbf{p}}_F. \quad (34)$$

The final optimal weight coefficient vector is then given by

$$\mathbf{w}_{MT,opt} = \mathbf{w}_{R,opt} + j\mathbf{w}_{I,opt}. \quad (35)$$

Note, that as for the CS formulation discussed in the previous subsection the MT-BCS scheme detailed here is unable to guarantee an SSSTA as an outcome. This is because it is in effect finding a sparse weight coefficient vector without considering where the associated dipoles are located. As a result, it is possible that there could be multiple dipoles present at the optimised locations (optimised locations refers to the locations with one or more non-zero valued weight coefficients). This means the desired reduction in mutual coupling effects when implemented in practice will not be achieved. Instead to ensure an SSSTA the methods discussed in the following subsections should be considered.

#### D. Iterative Minimum Distance Sampling Method for SSSTAs

In the above two formulations, there is no way to ensure that an SSSTA is achieved. This is due to the fact that only the weight coefficients associated with a given dipole are minimised, rather than considering if there are any co-located dipoles.

To solve this problem it is proposed to extend the idea of imposing a physical size constraint on the optimisation from [17]. However, when directly applied these methods only ensure that there is a minimum distance between the optimised antenna locations. Therefore, in this instances they could not guarantee an SSSTA as there can potentially be three dipoles at each antenna location. As a result, it is necessary to also consider the fact that co-located dipoles at a given location can also be seen as breaking the minimum separation of a physical

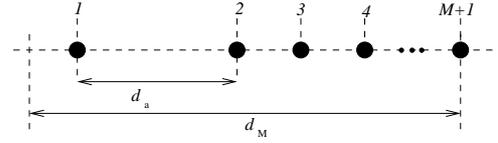


Fig. 2. Illustration of the iterative sampling method used, where  $d_a$  is the size constraint being applied,  $d_M$  the potential aperture of the array and  $\bullet$  indicates a dipole location.

size constraint. In this work we use the idea of the IMDSM and AIRMS algorithms proposed in [17] to ensure an SSSTA is achieved as the final solution.

Note, that the iterative nature of the IMSDM based approaches means that the relationship between  $M$  or  $\alpha$  and the algorithms performance becomes less predictable. Consider the fact that the value of  $M$  used affects where the first dipole is located. This then defines the remaining aperture, which is again sampled using  $M$  grid points. As a result the density of the sampling grid in the next iteration varies depending on where the previous dipole was placed and the value of  $M$ , which in turn makes it difficult to predict how the performance will be effected by  $M$ . The effects of  $\alpha$  can also be hard to predict for similar reasons.

1) *CS Based IMDSM*: To begin with, the full aperture of the array is uniformly sampled and an estimate of the weight coefficients found using (16), with the first cluster of dipoles that are too close together being merged to give the first location as shown in Figure 2. At this point if there are multiple dipoles at the merged location the least significant are discarded to leave a single dipole present. The remainder of the aperture is then uniformly sampled, ensuring that the next dipole will be at least the distance of the size constraint away. This process is then repeated until there is no room for further dipoles.

It is worth noting that this method has involved the merger of dipole locations and has the potential for some dipoles to be discarded in order to avoid co-located dipoles. As a result the weight coefficients may no longer be optimal for the given dipole locations and orientations. However, the locations and orientations can be used to efficiently implement a fixed beamformer, by minimising the sidelobe levels while keeping a unitary response for the mainlobe location. This is detailed below in Section II-D3.

2) *MT-BCS Based IMDSM*: In essence the same iterative procedure is followed in this instance. The initial set of weight coefficients used to find the first cluster is instead found using the MT-BCS procedure detailed in Section II-C. For subsequent iterations some changes have to be made to ensure that the method of solving the problem can account for the fact that some dipole locations and orientations have been fixed and will be contributing to the overall response.

As a result, consider the following

$$\check{\mathbf{p}}_R = \check{\mathbf{S}} \check{\mathbf{w}}_R, \quad \check{\mathbf{p}}_I = \check{\mathbf{S}} \check{\mathbf{w}}_I, \quad (36)$$

where  $\check{\mathbf{p}}_R$  and  $\check{\mathbf{p}}_I$  are found by subtracting the response due to the locations fixed in the previous iteration from the reference response in the previous iteration. Then from the

remaining uniformly sampled aperture in the current iteration we construct  $\tilde{\mathbf{S}}$  and the resulting estimate of the weight coefficients are given by  $\tilde{\mathbf{w}} = \tilde{\mathbf{w}}_{R,opt} + j\tilde{\mathbf{w}}_{I,opt}$ . Following the MT-BCS scheme detailed in Section II-C the solution is

$$\tilde{\mathbf{w}}_{F,opt} = \left( \text{diag}(\tilde{\mathbf{a}}_{opt}) + \tilde{\mathbf{S}}^T \tilde{\mathbf{S}} \right)^{-1} \tilde{\mathbf{S}}^T \tilde{\mathbf{p}}_F. \quad (37)$$

This process is repeated, with the merging and discarding of dipoles. As a result it is again necessary to use the method for redesigning the weight coefficients detailed below.

3) *Fixed Beamformer Design for Given Dipole Locations and Orientations*: After obtaining the dipole locations and orientations using the CS-IMDSM or BCS-IMDSM, it is necessary to re-design the coefficients of the array to provide a closer approximation to the desired responses. This is a classic fixed beamformer design problem and can be solved using the method described below, which is applicable to any arbitrary array geometry.

The redesign of the weight coefficients is achieved by minimising the sidelobe levels subject to a unitary response for the mainlobe direction. This can be formulated as

$$\begin{aligned} \min_{\tilde{\mathbf{w}}_{re}} \quad & \|\hat{\mathbf{p}}_r - \tilde{\mathbf{S}}(\hat{\mathbf{w}}_{mask} \circ \hat{\mathbf{w}}_{re})\|_2 \\ \text{subject to} \quad & R(\tilde{\mathbf{S}}_{ML}(\hat{\mathbf{w}}_{mask} \circ \hat{\mathbf{w}}_{re})) = 1 \\ & I(\tilde{\mathbf{S}}_{ML}(\hat{\mathbf{w}}_{mask} \circ \hat{\mathbf{w}}_{re})) = 0, \end{aligned} \quad (38)$$

where  $\hat{\mathbf{w}}_{mask} = [\mathbf{w}_{mask}, \mathbf{w}_{mask}]^T$  and  $\mathbf{w}_{mask}$  is a series of 1s and 0s to ensure only the correct dipole orientations are used,  $\hat{\mathbf{w}}_{re} = [R(\mathbf{w}_{re}), I(\mathbf{w}_{re})]^T$ ,  $\tilde{\mathbf{S}} = \begin{pmatrix} R(\mathbf{S}) & -I(\mathbf{S}) \\ I(\mathbf{S}) & R(\mathbf{S}) \end{pmatrix}$ ,  $\tilde{\mathbf{S}}_{ML}$  only considers the mainlobe direction and  $\circ$  denotes the Hadamard product.

#### E. Altered Iterative Reweighted Minimisation Scheme for SSSTAs

To avoid the merging and discarding of dipoles as required for IMDSM, this work also proposes an AIRMS. Here the reweighting scheme in (17) is adapted to also penalise dipole locations that are co-located or that are too close together [17]. This gives the following reweighting scheme

$$\delta_m^i = \begin{cases} (|w_m^{i-1}| + \epsilon)^{-1} & m = 1 \\ (|w_m^{i-1}| + \epsilon)^{-1} & m > 1 \text{ and constraint met} \\ (\epsilon)^{-1} & \text{otherwise.} \end{cases} \quad (39)$$

Now the iterative procedure is repeated until a solution that complies with the size constraint being enforced is obtained.

Unfortunately, this algorithm will not always guarantee a viable solution, due to the presence of  $\epsilon$  in the calculation of reweighting terms. The inclusion of  $\epsilon$  is required for numerical stability, but prevents a zero weight coefficient in the current iteration guaranteeing a zero weight coefficient in the next iteration. Based on the authors' experience with different design parameters, if a solution is possible it will usually be achieved in less than 10 iterations.

It is also hard to predict if a solution will be achieved, or the performance level achieved, based on the selection of  $M$ . This is as the choice of  $M$  greatly effects how likely we are

to get a solution that meets the size constraint value. It may be expected that increasing  $M$  should allow an improvement in the algorithms performance as it is more likely to get the optimal locations included on the sampling grid. This also makes it more likely that two or more dipoles will be located closer together than the size constraint making it harder to get a valid solution.

### III. DESIGN EXAMPLES

This section provides design examples to verify the effectiveness of the proposed methods. All examples are implemented on a computer with an Intel Xeon CPU E3-1271 (3.60GHz) and 16GB of RAM.

#### A. Broadside Example

Here a broadside design example will be considered to verify the effectiveness of the proposed methods. Note, two off-broadside design examples (which indicate the proposed methods can be effective for mainlobes over the entire angular range) has been omitted to conserve space but can be provided on request to interested readers.

Although the AIRMS does not necessarily require the weight coefficients to be redesigned, they have been here in order to allow a fairer comparison between all three design methods considered. Unless otherwise stated, the example considers the scenario of  $M = 301$  with a maximum possible aperture of  $10\lambda$ . For the MT-BCS based design method the values of  $\beta_{MT-1}$  and  $\beta_{MT-2}$  are set as suggested in [29], with the value of  $\check{\sigma}^2$  being found from the CS-IMDSM and AIRMS design examples. In this work the CS-IMDSM and AIRMS are implemented using cvx, a package for specifying and solving convex programs [37], [38].

Note, the selection of  $M$  has been made to get close to the sampling density suggested in [21], while also accounting for the fact that the proposed methods have to consider three antennas at each grid point rather than a single antenna. As discussed for the proposed methods it is also hard to predict how changing  $M$  will effect the performance of the algorithms (in the case of the AIRMS a solution is not even always guaranteed). Experience with different design examples suggest that  $M = 301$  for a  $10\lambda$  aperture usually ensures a suitable solution will be achieved by at least one of the three proposed methods.

For completeness the response from an equivalent ULA is also provided as a further comparison. To ensure optimised dipole locations and orientations for the ULAs, solve the minimisation in (38) with  $\mathbf{w}_{mask} = [1, 1, 1, \dots, 1]^T$  to allow the three dipole orientations at each location to be considered. Then a new  $\mathbf{w}_{mask}$  is constructed in order to keep only the most significant dipole orientations at each location. The minimisation in (38) is then resolved to give the final optimised dipole orientations and locations.

As a broadside design example is being considered the mainlobe is given by  $\theta_{ML} = 0^\circ$  for  $\phi_{ML} = 90^\circ$ , with the sidelobe regions defined by  $\theta_{SL} = [10^\circ, 90^\circ]$  for  $\phi_{SL} = \pm 90^\circ$  and being sampled every  $1^\circ$ . The polarisation information is



TABLE V  
Performance comparison for the broadside design examples.

Example	CS-IMDSM	BCS-IMDSM	AIRMS	ULA
Aperture/ $\lambda$	9.11	9.33	6.97	10
$\Delta d/\lambda$	0.91	0.93	0.87	0.50
Number of dipoles (% decrease)	11 48	11 48	9 57	21 0
Error	1.00	0.43	0.46	0.64
Amplitude of closest sidelobe (dB)	-20.02	-31.47	-30.55	-26.83
Computation time (seconds)	363.16	4.38	62.03	1.17
Number of iterations	11	11	3	2

TABLE VI  
Performance comparison for the CS-IMDSM broadside design examples.

M	101	201	301	401
Aperture/ $\lambda$	9.08	7.19	9.11	9.13
$\Delta d/\lambda$	0.91	0.90	0.91	0.91
Number of dipoles (% decrease)	11 48	10 52	11 48	11 48
Error	1.07	1.12	1.00	1.25
Amplitude of closest sidelobe (dB)	-14.18	-17.85	-20.02	-10.47
Computation time (seconds)	47.46	235.94	363.16	546.89
Number of iterations	11	10	11	11

increases with the problem size.

To illustrate the effects of the value of  $M$  used, now consider the same design example again with the values  $M = 101$ , 201 and  $M = 401$ , along with the original value of  $M = 301$ . The performance measures for the three proposed methods are summarised in Tables VI-VIII.

As expected, increasing the value of  $M$  has increased the computation for the three proposed design methods. This is because the design methods now consider a larger sampling grid for each iteration, which in turn means a longer computation time. However, the effect on the other performance measures used has proven to be harder to predict.

For each of the design methods varying  $M$  can alter the aperture of the designed array and the dipoles required to implement it in practice. The mean adjacent dipole separation

TABLE VII  
Performance comparison for the BCS-IMDSM broadside design examples.

M	101	201	301	401
Aperture/ $\lambda$	9.05	9.49	9.33	8.88
$\Delta d/\lambda$	0.91	0.95	0.93	0.89
Number of dipoles (% decrease)	11 48	11 48	11 48	11 48
Error	0.82	0.87	0.43	0.81
Amplitude of closest sidelobe (dB)	-22.41	-20.56	-31.47	-19.63
Computation time (seconds)	3.43	2.86	4.38	38.00
Number of iterations	11	11	11	11

TABLE VIII  
Performance comparison for the AIRMS broadside design examples.

M	101	201	301	401
Aperture/ $\lambda$	NA	6.95	6.97	6.98
$\Delta d/\lambda$	NA	0.87	0.87	0.87
Number of dipoles (% decrease)	NA	9 57	9 57	9 57
Error	NA	0.48	0.46	0.45
Amplitude of closest sidelobe (dB)	NA	-24.61	-30.55	-29.88
Computation time (seconds)	NA	34.06	62.03	99.06
Number of iterations	NA	2	3	2

TABLE IX  
Performance comparison for varying values of  $\alpha$ .

$\alpha$ (method)	0.35 (CS-IMDSM)	0.65 (CS-IMDSM)	0.65 (AIRMS)
Aperture/ $\lambda$	5.31	8.89	6.10
$\Delta d/\lambda$	0.88	0.89	0.87
Number of dipoles (% decrease)	7 67	11 48	8 62
Error	1.33	0.66	0.63
Amplitude of closest sidelobe (dB)	-16.83	-21.22	-26.26
Computation time (seconds)	379.95	339.11	71.84
Number of iterations	8	11	2

has remained reasonably constant and for the CS-IMSDM method the smallest separation has even occurred for the largest value of  $M$ . However, for the design of traditional sparse arrays using CS-based methods, increasing the value of  $M$  would lead to an expected increase in the mean adjacent dipole separation. This is because a denser grid will be able to give a closer approximating to the ideal locations and as a result uses less dipole in total. By looking at the error between the designed responses and the ideal response, along with the amplitudes of the closest sidelobes, it can be seen that the effect on the desirability of the designed response is similarly hard to predict in advance.

Finally, now consider the effect of  $\alpha$  on the performance of the CS-IMDSM and AIRMS for the broadside design example. Two further values of  $\alpha$  will be considered,  $\alpha = 0.35$  and 0.65, respectively. The performance of the two methods for these values is summarised in Table IX. For traditional CS based problems it would be expected to see that increasing the value of  $\alpha$  would increase the amount of error allowed, thus allowing extra sparsity to be introduced. However, here we can see the iterative nature of the algorithms has made predicting the effects of  $\alpha$  difficult. Note, the reason why no results are shown for AIRMS with  $\alpha = 0.35$  is that no solution was obtained in this case.

## B. Discussion

This subsection presents a discussion of the main results in light of the implications for optimal parameter selection. These points can be summarised as follows:

- 1) From the broadside design example it can be seen that increasing the value of  $M$  always increases the computation time as more grid points are being considered. For CS or BCS it would be reasonable to expect that increasing  $M$  would improve the solution in terms of sparsity and desirability of the achieved response. The iterative nature of the algorithms makes it harder to predict the effects on error between the reference and achieved responses and the number of dipoles required. Experience suggests that  $M = 301$  is the best tradeoff to make.
- 2) The iterative nature of the algorithms has also made it difficult to predict the effects of varying the value of  $\alpha$  for the CS-IMDSM and AIRMS. It is worth noting two points. Firstly a value of  $\alpha = 0$  would mean that the approximation of the reference pattern would have to be exact. This is unlikely to be possible when the ideal response is used. Secondly, a value of  $\alpha = 1$  will result in a response of all zeros and no dipoles being used, as  $\|[0, 0, \dots, 1, \dots, 0, 0]^T - [0, 0, \dots, 0, \dots, 0, 0]^T\|_2 = 1$ .
- 3) The value of  $L$  has to be large enough to consider all the angular and polarisation points of interest, as an acceptable response can not be guaranteed for the points not directly considered. Increasing  $L$  further when this has been achieved adds computational complexity for no further gain in desirability of the array's response.

It is also worth considering the problem of selecting which of the three proposed methods should be used in a given situation. There are 4 criteria to be considered: guarantees of a solution, the sparsity introduced, error between the reference and designed responses and the computational efficiency. These 4 points are now considered in turn and recommendations made about which method to use.

- 1) Guarantees of a solution: The results provided show that the AIRMS was the only one not to always give a solution. This would suggest using one of the other two methods when guarantees of a solution is the overriding factor. The selection of which of the remaining two methods should be used depends on which of the remaining criteria are prioritised.
- 2) The sparsity introduced: The results given above indicate that the CS-IMDSM tends to give the sparsest solution (followed by the BCS-IMDSM and then AIRMS) so should be selected when this criterion is the most important.
- 3) Error between reference and designed responses: In terms of the amount of error between the reference and designed responses the BCS-IMDSM has been shown to give the best performance. This can be explained by the fact that the less dipoles used the more error is expected and the BCS-IMDSM method had lower levels of sparsity than the CS-IMDSM (while the AIRMS was not always guaranteed to give a solution).
- 4) The computational efficiency: If computational efficiency is prioritised over the other criteria the authors would suggest considering the BCS-IMDSM as the results consistently show it is the most efficient method

(followed by AIRMS, when it gives a solution, and CS-IMDSM).

#### IV. CONCLUSIONS

In this work the problem of designing sparse SSSTA has been addressed for the first time. Novel CS and BCS based approaches have been proposed to solve the problem of simultaneously optimising dipole locations and orientations, with a minimum spacing being used to avoid co-located dipoles. Design examples have been provided and show that an accurate approximation of a reference pattern can be achieved using fewer dipoles than a comparable uniform SST array (48%-67% reduction in the number of dipoles). This work has focused on the design of linear SSSTAs for a single signal polarisation of interest. In order to fully control a wide range of signal polarisations a planar array may be necessary. Extending the proposed approaches to this case is seen as an area for future research.

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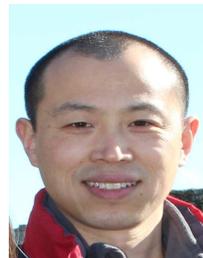
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