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Distributed predictive control with minimization of mutual disturbances☆

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Abstract

In this paper, a distributed model predictive control scheme is proposed for linear, time-invariant dynamically coupled systems. Uniquely, controllers optimize state and input constraint sets, and exchange information about these—rather than planned state and control trajectories—in order to coordinate actions and reduce the effects of the mutual disturbances induced via dynamic coupling. Mutual disturbance rejection is by means of the tube-based model predictive control approach, with tubes optimized and terminal sets reconfigured on-line in response to the changing disturbance sets. Feasibility and exponential stability are guaranteed under provided sufficient conditions on non-increase of the constraint set parameters.

Keywords: decentralization; time-invariant; control of constrained systems; optimization-based controller synthesis; parametric optimization.

1. Introduction

Model Predictive Control (MPC) has become one of the most popular advanced control techniques [1], with many industrial applications [2] and mature theoretical foundations [3]. The key to this success is the inherent flexibility of MPC, which allows for complex issues such as constraints or delays to be dealt with explicitly, when otherwise the off-line determination of a control law would be prohibitively difficult. Despite this, the control of large-scale, interconnected or networked systems—such as chemical plants [4], electricity networks [5] or teams of vehicles [6]—still presents significant difficulties to MPC [7]. For example, the organizational structure of the system—and its information flows—may not be conducive to a centralized control approach. Moreover, even if it is, the MPC optimization problem for the whole system may be too large to solve within the required time.

For this reason, significant attention has been been given in the past decade to *distributed* forms of model predictive control (DMPC) [8–10]. In DMPC, the optimal control problem is decomposed into several smaller sub-problems that are distributed to a set of local controllers or *control agents*. Each controller or agent is responsible for controlling a subsystem composed of a subset of the system states and control inputs. In order to achieve system-wide stability and satisfactory closed-loop performance, the agents exchange information so that they can coordinate their decision making. Many schemes have been proposed to date, and differ according to the particularities of the scenarios in which they are applied: for example, the way in which the system is decomposed, the source of coupling, or the limits in the communication or computation capacity [10].

One of the fundamental, and most researched, problems in DMPC is control of linear time-invariant systems coupled via dynamics. The problem is non-trivial since the states and inputs of one subsystem affect others too, leading to mutual disturbances; hence, coordination is usually needed to ensure satisfactory performance of the overall system. Many approaches have been proposed [8–10], and almost all involve the sharing of planned control sequences or state trajectories between controllers. Recently, attention has focused on tube MPC [11] as a means for rejecting the mutual disturbances arising from these subsystem interactions. The first tube-based DMPC approaches [12, 13] were developed for dynamically *decoupled*, uncertain subsystems with coupled constraints; each controller uses the tube technique to reject bounded local disturbances. The direct application of that approach to systems with dynamic coupling will, however, result in excessive conservativeness, since the bounded disturbance set for each subsystem must account for *all* possible state and input interactions (and not just, for example, deviations of neighbours' states and inputs from planned, or reference, trajectories). To circumvent this, improved proposals have been made: in [14], tube-based controllers share reference trajectories and maintain true states and inputs in bounded neighbourhoods of these. In [15], the tube MPC concept is applied *twice* by each controller: once to maintain a planned perturbed state trajectory around a planned nominal trajectory, then again to maintain the true, perturbed state trajectory around the planned one.

Though providing a natural route to guaranteed feasibility and stability, a key drawback of the tube-based approaches is conservatism because, ultimately, the mutual disturbance induced by state and/or input coupling has to be bounded. If the

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state and input constraint sets are large, then this naturally leads to large disturbance sets and, hence, more tightly constrained local optimal control problems, even for [14, 15]. In this paper, we attempt to overcome this drawback by exploiting the fact that, often, subsystems do not *use* all of their state and input constraint sets and, hence, the mutual disturbance sets can be reduced by considering this. The main technical development is that local controllers, when solving their optimal control problems, optimize not only the control sequence but also the *sizes* of the state and input constraint sets. In other words, subsystem state and input sets are contracted to the smallest sizes sufficient to meet control objectives, which in turn leads to smaller disturbance sets. Controllers then share information about these state and input sets—rather than planned state and control trajectories—in order that they may compute a smaller estimate of the set of possible disturbances. Finally, to reject these bounded disturbances, the tube MPC technique [11] is applied. However, in this paper, the disturbance invariant sets required for tube MPC are optimized online to take into account the changing sizes of the disturbance sets.

The sharing of *sets* of states and inputs has similarities with the "contract-based" DMPC approach [16], wherein subsystems share "contract sets" about their future behaviour, based on reachable sets computed at each time step given current knowledge of uncertainty. Our work differs in several details, including (i) the use of decoupled positively invariant sets as terminal conditions, which are less complex objects, and easier to compute, than the inter-dependent robust invariant sets required in [16]; (ii) in our approach, the complexity of each MPC problem is similar to conventional MPC, and the shared information between subsystems is of parameterized versions of the state and input constraint sets, which are readily available, while in [16] sequences of reachable sets are required to be computed within each MPC optimization; (iii) we offer a comprehensive way to compute the required disturbance sets and robust invariant sets that arise from the shared state and input sets, via a single linear program (LP).

This latter aspect, in particular, of the proposed approach also leads to similarities with the "plug-and-play" approach to decentralized MPC [17]. In that approach, subsystem controllers re-compute disturbance invariant sets on-line in order to account for changes to disturbance sets. However, there are two key differences: firstly, in [17], only the effect of adding or removing subsystems from the overall system is considered when disturbance sets are re-computed, while in this paper we re-compute disturbance sets to account for how much of the constraint sets planned state and input trajectories are using. Secondly, in [17] the notion of robust control invariant (RCI) sets [18] is used: each subsystem controller solves an LP to compute an RCI set and an associated feedback control law which are then used as, respectively, the tube cross-section set and tube controller. In this paper, however, we retain the original notion in tube MPC of robust positively invariant (RPI) sets: each controller retains the same (linear) tube controller throughout, but solves an LP to re-compute its RPI tube cross-section set to take into account changes to the mutual disturbance set. This is achieved by exploiting a recently developed method for computing, via a single LP, an RPI set characterized by *a-priori* known inequalities [19]; we make a further extension to this approach to include the computation of the disturbance set (which depends on neighbouring subsystems' states and inputs) implicitly in the RPI set optimization, removing the need to compute the disturbance set explicitly beforehand.

A preliminary version of this paper appeared in [20], presenting the initial idea and results. In the current paper, the following additional contributions are made:

- A reconfigurable, parametric terminal set is designed, replacing the simple choice of the origin used in [20]. This set, which enlarges the region of attraction and improves closed-loop performance, adjusts automatically (on-line) to account for the changes in size and shape of the constraint sets.
- The ancillary on-line operations to re-compute disturbance invariant sets are refined and improved: RPI sets are computed directly from shared information, via a single LP, removing the need to explicitly construct disturbance sets via Minkowski summations as in [20]. Furthermore, the algorithm is generalized to permit re-configuration of sets at a lower rate than the main sampling rate, in order to reduce the on-line computational burden. Further simplifications are described and discussed, including a scalar implementation of the algorithm that requires minimal on-line computation in addition to the MPC problem.

The paper is organized as follows. Preliminary details and the problem statement are given in Section 2. In Section 3, the distributed optimal control problem, including the parametric design of the terminal set, is presented. The distributed control algorithm is defined in Section 4, together with details and explanations of on-line computations. Theoretical guarantees of recursive feasibility and stability, under the sufficient condition of non-increase of the state and input constraint set parameters, are established in Section 5. In Section 6, simulations of the algorithm are presented for an example system, before concluding remarks are made in Section 7.

Notation: The sets of non-negative and positive reals are denoted, respectively, $\mathbb{R}_{0+}$ and $\mathbb{R}_{+}$. The notation $[a, b]^n$ means the n-dimensional product set $[a, b] \times [a, b] \times \cdots \times [a, b]$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$. For $a, b \in \mathbb{R}^n$, $a \leq b$ applies element by element. The ball of radius δ is $\mathcal{B}(\delta)$; the dimension will be clear from the context. The distance of a point $x \in \mathbb{R}^n$ from a set $X \subset \mathbb{R}^n$ is $|x|_X \triangleq \inf_{y\in X} |x - y|$. AX denotes the image of a set $X \subset \mathbb{R}^n$ under the linear mapping $A : \mathbb{R}^n \mapsto \mathbb{R}^p$, and is given by $\{Ax : x \in X\}$. For $X, Y \subset \mathbb{R}^n$, the Minkowski sum is $X \oplus Y \triangleq \{x + y : x \in X, y \in Y\}$; for $Y \subset X$, the Minkowski difference is $X \ominus Y \triangleq \{x \in \mathbb{R}^n : Y \oplus \{x\} \subset X\}$. For $X \subset \mathbb{R}^n$ and $a \in \mathbb{R}^n$, $X \oplus a$ means $X \oplus \{a\}$. The support function of a set $X \subset R^n$ evaluated at $y \in \mathbb{R}^n$ is $h(X, y) \triangleq \sup\{y^\top x : x \in X\}$. A polyhedron is an intersection of a finite number of halfspaces, and a polytope is a closed and bounded polyhedron. Unless otherwise indicated, a subscript i denotes a variable or parameter of subsystem i. The column vectors of zeros and ones are denoted **0** and **1** respectively, the length of which will be clear from the context.

2. Preliminaries

In this section, the problem statement and some necessary preliminary details are presented.

2.1. System dynamics and structure

We consider the discrete-time, linear time-invariant system

$$x^+ = Ax + Bu,$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state and input, and x^+ is the successor state. The system is partitioned into a set $\mathcal{N} = \{1, \ldots, M\}$ of subsystems, each described as

$$x_i^+ = A_{ii}x_i + B_{ii}u_i + \sum_{j\in\mathcal{N}_i}(A_{ij}x_j + B_{ij}u_j), \tag{1}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$ are the state and input of subsystem $i \in \mathcal{N}$, with $x = (x_1, \ldots, x_M)$ and $u = (u_1, \ldots, u_M)$ being the corresponding aggregated state and input vectors, and $A_{ij} \in \mathbb{R}^{n_i\times n_j}$, $B_{ij} \in \mathbb{R}^{n_i\times m_j}$. The latter are used to define the set of *neighbours* of subsystem i as

$$\mathcal{N}_i \triangleq \{j \in \mathcal{N} \setminus \{i\} : [A_{ij}\ B_{ij}] \neq 0\}.$$

Assumption 1. Each (A_{ii}, B_{ii}), $i \in \mathcal{N}$ is stabilizable.

2.2. Constraints

Each subsystem $i \in \mathcal{N}$ is subject to local constraints,

$$x_i \in \mathbb{X}_i \qquad\qquad u_i \in \mathbb{U}_i.$$

Assumption 2. $\mathbb{X}_i$ and $\mathbb{U}_i$ are polytopes and each contains the origin in its interior.

In particular, let $\mathbb{X}_i \triangleq \mathbb{X}_i(\mathbf{1})$ and $\mathbb{U}_i \triangleq \mathbb{U}_i(\mathbf{1})$, where

$$\mathbb{X}_i(a_i) \triangleq \{x_i \in \mathbb{R}^{n_i} : C_i^x x_i \leq a_i\}, \forall a_i \in \mathbb{R}_{0+}^{r_i^x},$$
$$\mathbb{U}_i(b_i) \triangleq \{u_i \in \mathbb{R}^{m_i} : C_i^u u_i \leq b_i\}, \forall b_i \in \mathbb{R}_{0+}^{r_i^u},$$

i.e., polytopic sets of r_i^x and r_i^u linear inequalities respectively. $\mathbb{X}_i(\mathbf{1})$ and $\mathbb{U}_i(\mathbf{1})$ are the original, hard constraint sets, but, in this paper, we will assume that, in general,

$$x_i \in \mathbb{X}_i(a_i) \qquad\qquad u_i \in \mathbb{U}_i(b_i) \tag{2}$$

for some $a_i \in \mathbb{R}_{0+}^{r_i^x}$ and $b_i \in \mathbb{R}_{0+}^{r_i^u}$. That is, x_i and u_i belong to polytopes with the same normal vectors as $\mathbb{X}_i(\mathbf{1})$ and $\mathbb{U}_i(\mathbf{1})$ but different right-hand sides. Note that if $a_i \leq \mathbf{1}, b_i \leq \mathbf{1}$ then the original constraints are satisfied.

2.3. Local subsystem disturbances and invariance

The dynamics of subsystem i may be written

$$x_i^+ = A_{ii}x_i + B_{ii}u_i + w_i, \tag{3}$$

where w_i is a disturbance given by

$$w_i = \sum_{j\in\mathcal{N}_i}(A_{ij}x_j + B_{ij}u_j).$$

Given the constraints (2), the disturbance is bounded as

$$w_i \in \mathbb{W}_i \triangleq \bigoplus_{j\in\mathcal{N}_i}(A_{ij}\mathbb{X}_j \oplus B_{ij}\mathbb{U}_j).$$

Owing to the properties of $\mathbb{X}_i$ and $\mathbb{U}_i$, and linearity, $\mathbb{W}_i$ is a polytope with $0 \in \mathbb{W}_i$. Without loss of generality, we define $\mathbb{W}_i$ as a polytope constructed from r_i^w inequalities:

$$\mathbb{W}_i \triangleq \mathbb{W}_i(g_i) \triangleq \{w_i \in \mathbb{R}^{n_i} : C_i^w w_i \leq g_i\},$$

where $g_i \in \mathbb{R}_{0+}^{r_i^w}$, and, furthermore, we define the C_i^w in such a way that $\mathbb{W}(\mathbf{1})$ is the set formed from the originally sized constraint sets:

$$\mathbb{W}_i(\mathbf{1}) = \bigoplus_{j\in\mathcal{N}_i} A_{ij}\mathbb{X}_j(\mathbf{1}) \oplus B_{ij}\mathbb{U}_j(\mathbf{1}). \tag{4}$$

Finally, by Assumption 1, there exists a K_i such that $(A_{ii} + B_{ii}K_i)$ has all of its eigenvalues strictly within the unit circle. Given $\mathbb{W}_i$ and K_i, there also exists a robust positively invariant (RPI) set, $\mathcal{R}_i$, for each i, which satisfies the following definition:

$$(A_{ii} + B_{ii}K_i)\mathcal{R}_i \oplus \mathbb{W}_i \subseteq \mathcal{R}_i. \tag{5}$$

Assumption 3. For each $i \in \mathcal{N}$, given $\mathbb{W}_i = \mathbb{W}_i(g_i)$ and K_i there exists a polytope $\mathcal{R}_i = \mathcal{R}_i(q_i)$ satisfying (5), where

$$\mathcal{R}_i(q_i) \triangleq \{x_i \in \mathbb{R}^{n_i} : C_i^{\mathcal{R}} x_i \leq q_i\},$$

and $q_i \in \mathbb{R}_{0+}^{r_i^{\mathcal{R}}}$.

That is, we assume that the RPI set is polytopic and may be represented by $r_i^{\mathcal{R}}$ inequalities. In general, the size of $\mathcal{R}_i(q_i)$ depends on the size of $\mathbb{W}_i(g_i)$, which in turn depends on $\mathbb{X}_j(a_j)$, $\mathbb{U}_j(b_j)$ for $j \in \mathcal{N}_i$. However, analogous to the definition of $\mathbb{W}_i(\mathbf{1})$, we normalize $\mathcal{R}_i(q_i)$ so that $\mathcal{R}_i(\mathbf{1})$ is defined as the polytopic RPI set associated with the originally sized disturbance set $\mathbb{W}_i(\mathbf{1})$:

$$(A_{ii} + B_{ii}K_i)\mathcal{R}_i(\mathbf{1}) \oplus \mathbb{W}_i(\mathbf{1}) \subseteq \mathcal{R}_i(\mathbf{1}).$$

The following assumption is common in tube-based MPC [21], and limits the size of the disturbance set with respect to the state and input constraint sets. Here, it is effectively a limit on the strength of couplings.

Assumption 4. For all $i \in \mathcal{N}$, $\mathcal{R}_i(\mathbf{1}) \subseteq \text{interior}(\mathbb{X}_i(\mathbf{1}))$ and $K_i\mathcal{R}_i(\mathbf{1}) \subseteq \text{interior}(\mathbb{U}_i(\mathbf{1}))$.

Assumptions 1–4 are supposed to hold throughout.

2.4. Control objective

The control objective is to regulate the state of each subsystem to the origin while satisfying all constraints and minimizing the infinite-horizon, system-wide cost

$$\sum_{k=0}^{\infty} \sum_{i \in \mathcal{N}} \ell_i(x_i(k), u_i(k)), \tag{6}$$

where $\ell_i(x_i, u_i) \triangleq (1/2)(x_i^\top Q_i x_i + u_i^\top R_i u_i)$ and Q_i, R_i are positive definite matrices.

3. Distributed optimal control problem

In this section, the distributed optimal control problem, used in the proposed DMPC algorithm, is presented. First we present a standard optimal control problem for a subsystem, based on a conventional tube MPC approach [11, 21] albeit in a distributed setting: that is, we propose to control the subsystem (3) via the control policy

$$u_i = v_i + K_i(x_i - z_i)$$

where x_i is the current state of i, (z_i, v_i) are the current state and input of the *nominal* subsystem $z_i^+ = A_{ii} z_i + B_{ii} v_i$ (*i.e.*, neglecting interactions), and v_i is obtained from an MPC optimization employing this nominal model.

We also introduce a key difference with respect to conventional tube MPC: the constraint sets and RPI set are, respectively, the sets $\mathbb{X}_i(a_i)$, $\mathbb{U}_i(b_i)$ and $\mathcal{R}_i(q_i)$, which are parameterized by a_i, b_i and q_i, rather than the usual fixed sets. Subsequently, we modify this optimal control problem to include optimization of the state and input set parameters a_i and b_i, leaving q_i as a parameter, paving the way for DMPC with minimization of mutual disturbance sets.

3.1. Conventional tube-based distributed optimal control problem

At nominal state z_i, the parametric optimal control problem for subsystem i is

$$\bar{\mathbb{P}}_i(z_i; a_i, b_i, q_i)\colon \ \min_{\mathbf{v}_i} \{ V_i(z_i, \mathbf{v}_i) : \mathbf{v}_i \in \mathcal{V}_i(z_i; a_i, b_i, q_i) \}$$

where $\mathbf{v}_i$ is the sequence of controls to optimize

$$\mathbf{v}_i = \{v_i(0), \ldots, v_i(N-1)\},$$

the set $\mathcal{V}_i(z_i; a_i, b_i, q_i)$ is defined by the constraints

$$z_i(j+1) = A_{ii} z_i(j) + B_{ii} v_i(j),\ j = 1 \ldots N-1, \tag{7a}$$
$$z_i(0) = z_i, \tag{7b}$$
$$z_i(j) \in \mathbb{X}_i(a_i) \ominus \mathcal{R}_i(q_i),\ j = 0 \ldots N-1, \tag{7c}$$
$$v_i(j) \in \mathbb{U}_i(b_i) \ominus K_i \mathcal{R}_i(q_i),\ j = 0 \ldots N-1, \tag{7d}$$
$$z_i(N) \in \mathbb{X}_i^f(a_i, b_i; q_i). \tag{7e}$$

The cost V_i is a finite-horizon approximation to i's share of (6):

$$V_i(z_i, \mathbf{v}_i) = V_i^f(z_i(N)) + \sum_{j=0}^{N-1} \ell_i(z_i(j), v_i(j)),$$

where ℓ_i was previously defined, and the terminal cost V_i^f will—together with the terminal set $\mathbb{X}_i^f(a_i, b_i; q_i)$—be defined in Section 3.3.

In this problem, because the nominal dynamics are used for predictions, *i.e.*, without the perturbing effect of the coupled dynamics, then the state and control constraint sets are tightened to account for the ensuing prediction mismatch.

Denoting a feasible solution to the problem as $\mathbf{v}_i^*(z_i)$, whose existence is discussed at the end of this section, the control applied to the subsystem (3) is then

$$u_i^* = \kappa_i(x_i, z_i) = \bar{\kappa}_i(z_i) + K_i(x_i - z_i). \tag{8}$$

where $\bar{\kappa}_i(z_i)$ is the first control in the optimized sequence $\mathbf{v}_i^*(z_i)$. The second, linear feedback term is intended to reduce mismatch between the nominal and perturbed trajectories.

Finally, note that the domain of the value function, and therefore the control law, is

$$\bar{\mathcal{Z}}_i(a_i, b_i, q_i) \triangleq \{z_i \in \mathbb{R}^{n_i} : \mathcal{V}_i(z_i; a_i, b_i, q_i) \neq \emptyset\},$$

which is parameterized by a_i, b_i and q_i; the role of these parameters is discussed in the next section. By definition, a feasible solution to $\bar{\mathbb{P}}_i(z_i; a_i, b_i, q_i)$ exists if and only if $z_i \in \bar{\mathcal{Z}}_i(a_i, b_i, q_i)$; it is possible to characterize and compute the latter set (for given a_i, b_i and q_i) using standard methods [21].

3.2. Modified distributed optimal control problem

The application of the control law (8) under the assumption that $\mathbb{X}_i(a_i) = \mathbb{X}_i(\mathbf{1})$, $\mathbb{U}_i(b_i) = \mathbb{U}_i(\mathbf{1})$ and $\mathcal{R}_i(q_i) = \mathcal{R}_i(\mathbf{1})$ results in a straightforward specialization of tube MPC [11] to the M-subsystem system: the tube sets, and corresponding tightened constraint sets, are fixed and computed offline. It is simple to show (see, for example, [21, Ch. 3]) that if $\mathcal{R}_i(\mathbf{1}) \subset \mathbb{X}_i(\mathbf{1})$, $K_i \mathcal{R}_i(\mathbf{1}) \subset \mathbb{U}_i(\mathbf{1})$ and $z_i(0) = x_i(0) \in \bar{\mathcal{Z}}_i(\mathbf{1}, \mathbf{1}, \mathbf{1})$, then recursive feasibility and stability of the system is guaranteed. A consequence of this kind of robust approach is that no communication is needed between controllers; therefore, the control architecture is decentralized.

In this paper, however, the sets $\mathbb{X}_i(a_i)$, $\mathbb{U}_i(b_i)$ and $\mathcal{R}_i(q_i)$ will be allowed to vary over time (but not over the prediction horizon) by virtue of permitting the parameters a_i, b_i and q_i to vary. In particular, we will make use of a modified optimal control problem, in which the a_i and b_i that parameterize the state and input sets are now decision variables:

$$\mathbb{P}_i(z_i; q_i)\colon \ \min_{(\mathbf{v}_i, a_i, b_i)} V_i(z_i, \mathbf{v}_i) + \rho_a \|a_i\|_1 + \rho_b \|b_i\|_1$$
$$\text{subject to } \mathbf{v}_i \in \mathcal{V}_i(z_i, a_i, b_i; q_i), (a_i, b_i) \in [0,1]^{r_i^x} \times [0,1]^{r_i^u},$$

where $\rho_a > 0$ and $\rho_b > 0$ are weighting parameters. The domain is

$$\mathcal{Z}_i(q_i) \triangleq$$
$$\{z_i : \exists (a_i, b_i) \in [0,1]^{r_i^x} \times [0,1]^{r_i^u} \text{ s.t. } \mathcal{V}_i(z_i, a_i, b_i; q_i) \neq \emptyset\}.$$

By definition, $\mathcal{Z}_i(q_i) \supseteq \bar{\mathcal{Z}}_i(a_i, b_i, q_i)$ given $(a_i, b_i) \in [0,1]^{r_i^x} \times [0,1]^{r_i^u}$.

The idea behind this problem is that, at the same time as optimizing the control sequence, the sizes of the sets $\mathbb{X}_i(a_i)$, $\mathbb{U}_i(b_i)$ are minimized. Recall that the disturbance seen by a subsystem i depends on the $\mathbb{X}_j(a_j)$ and $\mathbb{U}_j(b_j)$ of $j \in \mathcal{N}_i$, so smaller a_j and b_j lead to smaller disturbance sets. Note that the RPI set parameter q_i remains a parameter, rather than a variable, of the optimization. Its use will be described in Section 4.

Remark 1. Here, and in the sequel, we consider the most general case of permitting state and/or input coupling between subsystems, and therefore both a_i and b_i are decision variables in the optimization problem for subsystem i. Notwithstanding, the proposed approach may be specialized to more specific system structures by fixing the appropriate variables; for example, for state-only coupling between subsystems, a_i is retained as a variable and b_i is fixed to $\mathbf{1}$.

3.3. Parametric terminal set and cost design

A standard approach to guaranteeing recursive feasibility and closed-loop stability in MPC, without requiring an infinite horizon, is to employ a terminal cost function V_i^f along with a terminal constraint set $\mathbb{X}_i^f$ with specified properties [21]. Typically, and especially in the context of tube-based MPC, the terminal set $\mathbb{X}_i^f$ is assumed, or constructed, to be

(i) Positively invariant for the nominal dynamics $z_i^+ = A_{ii}z_i + B_{ii}v_i$ under some terminal control law $v_i = K_i^f z_i$, chosen to stabilize (A_{ii}, B_{ii}). In other words,

$$(A_{ii} + B_{ii}K_i^f)\mathbb{X}_i^f \subseteq \mathbb{X}_i^f. \tag{9}$$

(ii) Admissible with respect to the tightened state and input constraints. That is,

$$\mathbb{X}_i^f \subseteq \mathbb{X}_i(a_i) \ominus \mathcal{R}_i(q_i) \tag{10a}$$

$$K_i^f\mathbb{X}_i^f \subseteq \mathbb{U}_i(b_i) \ominus K_i\mathcal{R}_i(q_i). \tag{10b}$$

The difficulty in the current setting is that the size and shape of the terminal set are restricted by the sizes and shapes of the constraint sets, which may change. While it is easy to satisfy these requirements with a simple terminal equality constraint (*i.e.*, $\mathbb{X}_i^f = \{0\}$), the design of a larger and less conservative set poses a non-trivial challenge, for the terminal set needs to be either recomputed or reconfigured as $\mathbb{X}_i(a_i)$ and $\mathbb{U}_i(b_i)$ change.

Reconfigurable terminal sets have been proposed in the context of setpoint, reference or target tracking and fault tolerant forms of MPC [22–26]; the idea is to parameterize an invariant, admissible set in terms of a steady-state target equilibrium pair, so that when the target changes the terminal set can be adjusted accordingly and automatically. In [27], a novel reconfigurable terminal set that is an inner approximation to the maximal admissible set (MAS) [28], parameterized by the right-hand side of the polytopic input constraint set (*i.e.*, here b_i), is proposed; the context is fault-tolerant control, wherein the failure of an actuator may be modelled as a change in the input constraint set.

Inspired by [27], the approach taken here is to design a reconfigurable terminal set that is parameterized by the state and input constraint vectors a_i and b_i. The following result assures the existence and properties of this set.

Lemma 1 (Parametric admissible invariant set). *Suppose K_i^f is such that $\Phi_i \triangleq A_{ii} + B_{ii}K_i^f$ has all eigenvalues strictly within the unit circle, and (a_i, b_i, q_i) are such that the sets $\mathbb{X}_i(a_i)\ominus\mathcal{R}_i(q_i)$ and $\mathbb{U}_i(b_i) \ominus K_i\mathcal{R}_i(q_i)$ are non-empty. Then the set*

$$\mathcal{O}_i^\infty(a_i, b_i; q_i) = \{z_i : E_i\Phi_i^k z_i \in \mathbb{Y}_i(a_i, b_i, q_i), k = 0, 1, \ldots\},$$

*where $E_i \triangleq [I \quad (K_i^f)^\top]^\top$ and $\mathbb{Y}_i(a_i, b_i, q_i) \triangleq (\mathbb{X}_i(a_i) \ominus \mathcal{R}_i(q_i)) \times (\mathbb{U}_i(b_i) \ominus K_i\mathcal{R}_i(q_i))$, is (i) compact, convex and contains the origin; (ii) constraint admissible and Φ_i-invariant (*i.e.*, $\mathbb{X}_i^f = \mathcal{O}_i^\infty$ satisfies* (9) *and* (10)*); (iii) finitely determined; (iv) inner approximated by the Φ_i-invariant polytope*

$$\{z_i : M_i^z z_i \leq c_i^z - M_i^{za}(a_i - s_i) - M_i^{zb}(b_i - t_i)\}. \tag{11}$$

Proof. Results (i)–(iii) follow directly by specializing the results of [28] to the setting of this paper. For (iv), consider the nominal subsystem dynamics augmented with the states of the constraint parameters, $\bar{a}_i \triangleq a_i - s_i$ and $\bar{b}_i \triangleq b_i - t_i$:

$$\begin{bmatrix} z_i^+ \\ \bar{a}_i^+ \\ \bar{b}_i^+ \end{bmatrix} = \begin{bmatrix} \Phi_i & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} z_i \\ \bar{a}_i \\ \bar{b}_i \end{bmatrix}, \tag{12}$$

where $s_{il} \triangleq h(\mathcal{R}_i(q_i), (C_{il}^x)^\top)$ is the lth element of s_i, for $l = 1 \ldots r_i^x$, and similarly $t_{il} \triangleq h(K_i\mathcal{R}_i(q_i), (C_{il}^u)^\top)$ for $l = 1 \ldots r_i^u$. Note that C_{il}^x (respectively C_{il}^u) corresponds to row l of the matrix $C_i^x \in \mathbb{R}^{r_i^x} \times \mathbb{R}^{n_i}$ (respectively $C_i^u \in \mathbb{R}^{r_i^u} \times \mathbb{R}^{m_i}$), so the transpose is taken to obtain a column vector.

The constraints that must hold for all times $k = 0 \ldots \infty$ are (10) and $\mathbf{0} \leq a_i \leq \mathbf{1}$, $\mathbf{0} \leq b_i \leq \mathbf{1}$. Using the polytopic definitions of $\mathbb{X}_i$ and $\mathbb{U}_i$,

$$\begin{bmatrix} C_i^x & -I & 0 \\ C_i^u K_i^f & 0 & -I \\ 0 & I & 0 \\ 0 & -I & 0 \\ 0 & 0 & I \\ 0 & 0 & -I \end{bmatrix} \begin{bmatrix} z_i \\ \bar{a}_i \\ \bar{b}_i \end{bmatrix} \leq \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix}$$

Owing to the neutrally stable $\bar{a}_i$ and $\bar{b}_i$ dynamics, the maximal constraint admissible set for (12) is not necessarily finitely determined. It is, however, if the constraints are inner-approximated as

$$\begin{bmatrix} C_i^x & -I & 0 \\ C_i^u K_i^f & 0 & -I \\ 0 & I & 0 \\ 0 & -I & 0 \\ 0 & 0 & I \\ 0 & 0 & -I \end{bmatrix} \begin{bmatrix} z_i \\ \bar{a}_i \\ \bar{b}_i \end{bmatrix} \leq \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} - \delta\mathbf{1} \\ \mathbf{0} + \delta\mathbf{1} \\ \mathbf{1} - \delta\mathbf{1} \\ \mathbf{0} + \delta\mathbf{1} \end{bmatrix} \tag{13}$$

where $0 < \delta < 1$ [28]. Then a finitely determined inner approximation to the maximal admissible set for the augmented dynamics (12) and constraint set (13) is

$$\begin{bmatrix} M_i^z & M_i^{za} & M_i^{zb} \\ 0 & M_i^a & 0 \\ 0 & 0 & M_i^b \end{bmatrix} \begin{bmatrix} z_i \\ \bar{a}_i \\ \bar{b}_i \end{bmatrix} \leq \begin{bmatrix} c_i^z \\ c_i^a \\ c_i^b \end{bmatrix}.$$

It follows that the parametric terminal set, which is constraint admissible and invariant, is given by

$$\{z_i : M_i^z z_i \leq c_i^z - M_i^{za}(a_i - s_i) - M_i^{zb}(b_i - t_i)\}.$$

and is valid for $\delta\mathbf{1} \leq a_i - s_i \leq \mathbf{1} - \delta\mathbf{1}$, $\delta\mathbf{1} \leq b_i - t_i \leq \mathbf{1} - \delta\mathbf{1}$. □

This inner approximation to the maximal constraint admissible set—parameterized by a_i and b_i—is employed as the terminal set $\mathcal{X}_i^f(a_i, b_i; q_i)$. Writing the terminal constraint (7e) in the form

$$M_i^z z_i(N) + M_i^{za} a_i + M_i^{zb} b_i \leq c_i^z + M_i^{za} s_i + M_i^{zb} t_i$$

where M_i^z, c_i^z, M_i^{za}, M_i^{zb}, s_i and t_i are parameters provided to the optimization, and $z_i(N)$, a_i and b_i are variables, the parametric dependence is clearly seen.

Remark 2 (Price of reconfigurability). The price of having a parametric, finite representation of a constraint admissible terminal set is complexity and conservativeness: the augmentation of the dynamics lifts the subsystem dynamics to a higher dimension and introduces eigenvalues on the unit circle (known to increase the finite determinedness index [28]), and while the tightening of constraints by δ is sufficient to guarantee finite determinedness of the set for the augmented system, it leads to an inner approximation of the maximal admissible set [28].

The remaining consideration is the terminal cost function. For this, we make the obvious choice of $V_i^f(z_i) = (1/2)z_i^\top P_i z_i$, where P_i is positive definite and satisfies

$$\Phi_i^\top P_i \Phi_i - P_i \leq -Q_i - (K_i^f)^\top R_i K_i^f.$$

4. Distributed control algorithm and implementation

The modified optimal control problem $\mathbb{P}_i(z_i; q_i)$ is used in the following algorithm.

Algorithm 1 (Distributed MPC for subsystem i).

Initial data: Sets $\mathbb{X}_i(\mathbf{1})$, $\mathbb{U}_i(\mathbf{1})$, $\mathbb{W}_i(\mathbf{1})$, $\mathcal{R}_i(\mathbf{1})$, $\mathbb{X}_i^f(a_i, b_i; q_i)$; matrices K_i and K_i^f, reconfiguration period T

Initialization: At $k = 0$, set $x_i = z_i = x_i(0)$, $q_i = \mathbf{1}$ and $p = 0$.

Online routine:

1. At time k and state (x_i, z_i), solve $\mathbb{P}_i(z_i; q_i)$ to obtain $v_i^* = \bar{\kappa}_i(z_i)$ and (a_i^*, b_i^*).
2. Apply $u_i = v_i^* + K_i(x_i - z_i)$ to subsystem i.
3. If $k = pT$,
   (a) Transmit a_i^*, b_i^* to subsystems $j \in \mathcal{N}_i$.
   (b) Compute $\mathbb{W}_i(g_i^+)$ and $\mathcal{R}_i(q_i^+)$.
   (c) Set $p = p + 1$.
4. Measure x_i^+ and compute $z^+ = A_{ii} z_i + B_{ii} v_i^*$.
5. Check if $x_i^+ - z_i^+ \in \mathcal{R}_i(q_i^+)$: if so, set $q_i = q_i^+$.
6. Set $(x_i, z_i) = (x_i^+, z_i^+)$, set $k = k + 1$, go to Step 1.

In this algorithm, initial data—including the sets $\mathbb{W}_i(\mathbf{1})$ and $\mathcal{R}_i(\mathbf{1})$—are provided in order that each subsystem controller may construct and solve its optimal control problem. The initial state of the nominal system is set equal to $x_i(0)$.

At time k, each subsystem solves its optimal control problem to obtain the nominal control input v_i for use in the control law of step 2. The optimization also minimizes the constraint set parameters, a_i and b_i. The idea is that, if subsystems are not "using" all of their state and constraint sets, then these can be contracted.

Every T steps, the optimized a_i^* and b_i^* are transmitted by controllers to neighbours (step 3a), so that each subsystem controller may use the received parameters to compute (in step 3b) its disturbance and RPI sets for use at the next time step. Following these computations, a check is made, in step 5, of the current states of the subsystem in regard to new RPI set. Before we present the details of the computational operations in step 3b, this checking step is explained.

4.1. Checking $x_i^+ - z_i^+ \in \mathcal{R}_i(q_i^+)$

Having received the vectors (a_j^*, b_j^*) from neighbours and computed $\mathbb{W}_i(g_i^+)$ and $\mathcal{R}_i(q_i^+)$, subsystem i then checks, in step 5, whether the successor states (x_i^+, z_i^+) satisfy $x_i^+ - z_i^+ \in \mathcal{R}_i(q_i^+)$. The rationale for this is to maintain recursive feasibility and constraint satisfaction guarantees despite changing the RPI set. In particular, and as will be shown in the next section, if $x_i^+ - z_i^+ \in \mathcal{R}_i(q_i^+)$ then it follows that $(A_{ii} + B_{ii}K_i)(x_i^+ - z_i^+) \oplus \mathbb{W}(g_i^+) \subseteq \mathcal{R}_i(q_i^+)$ and that the trajectory of (x_i, u_i) will satisfy all constraints. On the other hand, if $x_i^+ - z_i^+ \notin \mathcal{R}_i(q_i^+)$, then the same cannot be guaranteed; in that case, however, there exists the fail-safe option of using the current RPI set $\mathcal{R}_i(q_i)$, since $x_i^+ - z_i^+ \in \mathcal{R}_i(q_i)$. Because, as we will show, $\mathbb{W}_i(g_i^+) \subseteq \mathbb{W}_i(g_i)$ then $\mathcal{R}_i(q_i) \supseteq \mathcal{R}_i(q_i^+)$ and $(A_{ii} + B_{ii}K_i)(x_i^+ - z_i^+) \oplus \mathbb{W}(g_i^+) \subseteq \mathcal{R}_i(q_i)$. In other words, if the new RPI set does not meet the specified condition, the current RPI set can be used to maintain the guarantees of feasibility and stability. These properties of the controlled system will be established in Section 5.

Note that this checking step need only be performed every T steps, following the computation of a new $\mathcal{R}_i(q_i^+)$. For presentational convenience, however, it is presented in Algorithm 1 as part of the main routine.

4.2. Implementation: the polytopic case

In this section, implementation details of the algorithm—and particularly the ancillary computations in step 3b—are presented. First, it is shown how the disturbance sets may be computed from shared information via the solution of an LP. Having obtained the modified disturbance set, the new RPI set is obtained via applying the method in [19], which employs a single LP to compute an RPI set that is minimal with respect to the family of RPI sets represented by the same system of inequalities. Finally, it is shown how these two LPs may be combined, so that the RPI set can be computed directly from shared information, via a single LP, without the need to compute the disturbance set explicitly.

4.2.1. Computing $\mathbb{W}_i(g_i^+)$

In Step 3b, the new disturbance set is calculated using the optimized state and input set parameters (a_j^*, b_j^*) from neighbours. Moreover, as part of the initialization, the set $\mathbb{W}_i(\mathbf{1})$ must be computed and provided to the controller for subsystem i.

For given (a_j, b_j) for neighbours $j \in \mathcal{N}_i$, the disturbance set may be determined exactly as the finite Minkowski sum of polytopes

$$\bigoplus_{j\in\mathcal{N}_i} A_{ij}\mathbb{X}_j(a_j) \oplus B_{ij}\mathbb{U}_j(b_j). \tag{14}$$

The complexity of this polytope—and of the operation required to obtain it—depends on the state and input dimensions n_j and m_j, the number of halfspaces or vertices representing the state and input sets $\mathbb{X}_j(a_j)$ and $\mathbb{U}_j(b_j)$, and the number of neighbours in $\mathcal{N}_i$. More precisely, supposing the number of vertices of the polytope $A_{ij}\mathbb{X}_j(a_j) \oplus B_{ij}\mathbb{U}_j(b_j) \subset \mathbb{R}^{n_i}$ is d_j, then the number of vertices of the sum (14) is $O(\bar{d}^{n_i})$ where $\bar{d} = \max_{j\in\mathcal{N}_i} d_j$ [29], and the number of arithmetic operations to obtain it is $O(s)$ where $s = \prod_{j\in\mathcal{N}_i} d_j$ [30]. So, although polynomial in order, the complexity of the representation of the disturbance set could be high even for low dimensional ($n_i = 2$ or $n_i = 3$) subsystems. To mitigate this, we note that only an outer-approximation to the disturbance set is required:

$$\mathbb{W}_i(g_i) \supseteq \bigoplus_{j\in\mathcal{N}_i} A_{ij}\mathbb{X}_j(a_j) \oplus B_{ij}\mathbb{U}_j(b_j).$$

This justifies the assumption that the disturbance set be represented by r_i^w inequalities, where r_i^w can be chosen relatively small compared to the exact representation of the set, provided the above set inclusion holds. At time $k = 0$, the set $\mathbb{W}_i(\mathbf{1})$ is computed, using $a_j = \mathbf{1}$, $b_j = \mathbf{1}$, and provided to controller i.

For subsequent time steps, since each $\mathbb{X}_j(a_j)$ and $\mathbb{U}_j(b_j)$, for $j \in \mathcal{N}_i$, is defined by a constant number, respectively r_j^x and r_j^u, of inequalities, it follows that the (possibly outer-approximated) disturbance set may also be defined by a constant number, r_i^w, of inequalities regardless of the values of a_j and b_j (although some inequalities may, of course, be redundant for certain values). This has two significant implications: first, this motivates and justifies the use of an RPI set also defined by a constant, finite number of inequalities, as explained in the next subsection. More immediately, however, it implies that, when the (a_j, b_j) change to (a_j^*, b_j^*), the set $\mathbb{W}_i(g_i)$ is reconfigured to $\mathbb{W}_i(g_i^+)$ but retains the same complexity of representation. Therefore, the Minkowski summation need not be computed directly when the a_j and b_j change, and the new disturbance set can be computed via a more efficient means. In particular, note that the summation (14) may be re-written in terms of support functions

$$h(\mathbb{W}_i(g_i^+), w) = \sum_{j\in\mathcal{N}_i} h(A_{ij}\mathbb{X}_j(a_j^*), w) + h(B_{ij}\mathbb{U}_j(b_j^*), w)$$

for all $w \in \mathbb{R}^{n_i}$. To form $\mathbb{W}_i(g_i^+)$, given that we already have a representation $\mathbb{W}_i(g_i)$ as $\{w_i : C_i^w w_i \leq g_i\}$, it suffices to evaluate this summation for the vectors $(C_{i1}^w)^\top, \ldots, (C_{ir_i^w}^w)^\top$ that define the left-hand side of the inequality description of $\mathbb{W}_i$. Hence,

$$\begin{aligned} g_{il}^+ &= \sum_{j\in\mathcal{N}_i} h(A_{ij}\mathbb{X}_j(a_j^*), (C_{il}^w)^\top) + h(B_{ij}\mathbb{U}_j(b_j^*), (C_{il}^w)^\top) \\ &= \sum_{j\in\mathcal{N}_i} \max\{C_{il}^w A_{ij} x_j^l : x_j^l \in \mathbb{X}_j(a_j^*)\} \\ &\qquad + \max\{C_{il}^w B_{ij} u_j^l : u_j^l \in \mathbb{U}_j(b_j^*)\}. \end{aligned}$$

for each row l of g_i^+. This suggests that a sequence of LPs needs to be solved in order to determine g_i^+. However, further efficiencies can be made by combining these LPs into a single LP:

$$\max_{\substack{\{g_{il}^+, x_j^l, u_k^l\} \\ \forall l\in\{1,\ldots,r_i^w\}}} \sum_{l=1}^{r_i^w} g_{il}^+$$

subject to, for $j \in \mathcal{N}_i$ and $l = 1 \ldots r_i^w$,

$$\begin{aligned} g_{il}^+ &\leq \sum_{j\in\mathcal{N}_i} C_{il}^w (A_{ij} x_j^l + B_{ij} u_j^l), \\ C_j^x x_j^l &\leq a_j^*, \\ C_j^u u_j^l &\leq b_j^*. \end{aligned}$$

In this problem, $g_{il}^+ \in \mathbb{R}$, $x_j^l \in \mathbb{R}^{n_j}$ and $u_j^l \in \mathbb{R}^{m_j}$ are the decision variables for each $l = 1 \ldots r_i^w$.

4.2.2. Computing $\mathcal{R}_i(q_i^+)$ given $\mathbb{W}_i(g_i^+)$

The second operation required in Step 3b is the computation of the new RPI set associated with the latest disturbance set. Precisely, the problem is the compute $\mathcal{R}_i(q_i^+)$ for the closed-loop dynamics $x_i^+ \in (A_{ii} + B_{ii}K_i)x_i \oplus \mathbb{W}_i(g_i^+)$, where the latter is the updated disturbance set. We already assumed that, at all time steps, $\mathcal{R}_i(q_i)$ is a polytope defined by $r_i^{\mathcal{R}}$ inequalities and normal vectors $(C_{i1}^{\mathcal{R}})^\top, \ldots, (C_{ir_i^{\mathcal{R}}}^{\mathcal{R}})^\top$, where $C_{il}^{\mathcal{R}}$ is row l of $C_i^{\mathcal{R}}$, and we now justify this assumption.

Regarding the selection of $C_i^{\mathcal{R}}$ in order to define the set $\mathcal{R}_i(\mathbf{1})$ from $A_{ii} + B_{ii}K_i$ and $\mathbb{W}_i(\mathbf{1})$, there are two main issues: the size of the RPI set—the *minimal* RPI set is desirable to limit conservatism [11]—and the computational complexity of obtaining it, for Algorithm 1 requires it to be computed (or at least re-computed) on-line as $\mathbb{W}_i(g_i)$ changes. There are two obvious possibilities, but each with drawbacks.

- The mRPI is the smallest RPI set, and may be obtained via Minkowski summations. However, this set is finitely determined only if $(A_{ii} + B_{ii}K_i)^k\mathbb{W}_i = \beta\mathbb{W}_i$ for some $\beta \in [0, 1)$ (*e.g.*, for deadbeat K_i).
- A method exists for computing an ϵ-outer-approximation to the minimal RPI set to arbitrary accuracy [31], but requires the solving of an *a-priori* unknown (but finite) number of LPs and the Minkowski sum of an *a-priori* unknown (but finite) number of polytopes.

Neither is suitable for on-line use. As an alternative, therefore, we adopt the approach recently proposed in [19], based on the notions introduced in [32], which computes, via a single LP, the so-called (P, r)-mRPI set instead of the mRPI set. This is an RPI set defined by r pre-selected inequalities with left-hand side P, and which is minimal (smallest in volume) with respect to the family of RPI sets characterized by these same inequalities. In the context of this paper, solving the following LP computes, for some designer-specified $C_i^{\mathcal{R}}$, the right-hand side of the constraints q_i, in order for the set to be RPI for the disturbance set $\mathbb{W}_i(g_i)$. If such a q_i does not exist (which is the case iff the nullspace of $C_i^{\mathcal{R}}$ is not $(A_{ii} + B_{ii}K_i)$-invariant) then the LP is unbounded [19].

For the initial time, we suppose that the set $\mathcal{R}_i(\mathbf{1})$ has been designed off-line (*i.e.*, by employing the method of [31], or the one of [19] by some suitable choice of $C_i^{\mathcal{R}}$; examples are given in [19]). Then, at a subsequent time k, given $\mathbb{W}_i(g_i^+)$, the following LP computes an RPI set generated by the same number, $r_i^{\mathcal{R}}$, of inequalities and same normal vectors but right-hand side, q_i^+.

$$q_i^+ = c_i^* + d_i^* \text{ where } (c_i^*, d_i^*) = \arg \max_{\substack{\{c_{il}, d_{il}, \xi_i^l, \omega_i^l\} \\ \forall l \in \{1,\ldots,r_i^{\mathcal{R}}\}}} \sum_{l=1}^{r_i^{\mathcal{R}}} c_{il} + d_{il}$$

subject to, for all $l \in \{1, \ldots, r_i^{\mathcal{R}}\}$,

$$\begin{aligned} c_{il} &\leq C_{il}^{\mathcal{R}}(A_{ii} + B_{ii}K_i)\xi_i^l, \\ C_i^{\mathcal{R}}\xi_i^l &\leq c_i + d_i, \\ d_{il} &\leq C_{il}^{\mathcal{R}}\omega_i^l, \\ C_i^w\omega_i^l &\leq g_i^+. \end{aligned}$$

Then the set $\mathcal{R}_i(q_i^+)$ is RPI and—moreover—is the smallest RPI set defined by these $r_i^{\mathcal{R}}$ inequalities [19]. Further details and theoretical results may be found in [19].

4.2.3. *Computing $\mathcal{R}_i(q_i^+)$ without explicitly computing $\mathbb{W}_i(g_i^+)$*

A further simplification can be made to on-line operations by noting that the two previous optimization problems may be combined, leading to a direct way to compute the RPI set from shared data a_j^* and b_j^*. Thus,

$$q_i^+ = c_i^* + d_i^* \text{ where } (c_i^*, d_i^*) = \arg\max \sum_{l=1}^{r_i^{\mathcal{R}}} c_{il} + d_{il} \tag{15}$$

subject to, for $l = 1 \ldots r_i^{\mathcal{R}}$, $k = 1 \ldots r_i^w$ and $j \in \mathcal{N}_i$,

$$c_{il} \leq C_{il}^{\mathcal{R}}(A_{ii} + B_{ii}K_i)\xi_i^l, \tag{16a}$$
$$C_i^{\mathcal{R}}\xi_i^l \leq c_i + d_i, \tag{16b}$$
$$d_{il} \leq C_{il}^{\mathcal{R}}\omega_i^l, \tag{16c}$$
$$C_{ik}^w\omega_i^l \leq \sum_{j\in\mathcal{N}_i} C_{ik}^w(A_{ij}x_j^k + B_{ij}u_j^k), \tag{16d}$$
$$C_j^x x_j^k \leq a_j^*, \tag{16e}$$
$$C_j^u u_j^k \leq b_j^*. \tag{16f}$$

The decision variables of this problem are $c_i \in \mathbb{R}^{r_i^{\mathcal{R}}}$, $d_i \in \mathbb{R}^{r_i^{\mathcal{R}}}$, $\xi_i^l \in \mathbb{R}^{n_i}$ and $\omega_i^l \in \mathbb{R}^{n_i}$ for $l = 1 \ldots r_i^{\mathcal{R}}$, and $x_j^k \in \mathbb{R}^{n_j}$ and $u_j^k \in \mathbb{R}^{m_j}$ for $k = 1 \ldots r_i^w$ and $j \in \mathcal{N}_i$.

The outcome here is worth remarking upon: this procedure takes constraint sets from neighbours as inputs, and produces an RPI set by solving a single LP. This LP is the one that is solved on-line, in step 3b of Algorithm 1. The computational complexity and information requirements in order for controller i to formulate this problem are summarized next.

4.2.4. *On-line computational burden*

The algorithm specifies the solving of the optimal control problem $\mathbb{P}_i(z_i; q_i)$ at every sampling instant, and the ancillary LP every T steps. It should be noted that, owing to the robust feasibility and stability properties of tube-based MPC, which will be established in the next section under suitable sufficient conditions, it is not necessary to solve *any* problem at a given time step—including the optimal control problem—but it may be advantageous to do so for performance reasons (see [21, Ch. 3]). Nonetheless, it is important to evaluate the on-line computational complexity of the proposed approach.

The modified optimal control problem $\mathbb{P}_i(z_i; q_i)$ has similar complexity to a nominal LQ-MPC problem [11], but for the addition of $r_i^x + r_i^u$ additional non-negative variables for the parameterized constraints, and some additional inequalities to represent to parametric terminal set (how many depends on, *inter alia*, the choice of δ). Note that the modified problem is still a quadratic program (QP), despite the 1-norm cost on a_i and b_i, because the latter are constrained as non-negative and the problem may be written in the form

$$\min_{y_i}\{y_i^\top H_i y_i + f_i^\top y_i : G_i y_i \leq h_i\}$$

where $y_i = [\mathbf{v}_i^\top\ a_i^\top\ b_i^\top]^\top$ and H_i is positive semi-definite.

The ancillary LP (15) subject to (16), used to determine the RPI set, comprises $2(1 + n_i)r_i^{\mathcal{R}} + (n_i + m_i)r_i^w$ variables and $(2 + r_i^{\mathcal{R}})r_i^{\mathcal{R}} + r_i^w(\sum_{j\in\mathcal{N}_i} r_j^x + r_j^u)$ constraints. In order that controller i can formulate and solve this problem, it needs knowledge of the constraint matrices C_j^x and C_j^u for each subsystem $j \in \mathcal{N}_j$; these can be provided either initially, or transmitted at the same time as (a_j^*, b_j^*).

4.3. *Simplified implementation: the scaled set case*

Significant simplifications can be made to the optimal control problems and algorithm if, rather than allowing polytopic reconfiguration of the sets $\mathbb{X}_i(a_i)$ and $\mathbb{U}_i(b_i)$, the re-sizing of $\mathbb{X}_i$ and $\mathbb{U}_i$ (or just one of these, depending on system coupling – see Remark 1) is restricted to a simple scaling. Suppose $a_i = \alpha_i\mathbf{1}$ and $b_i = \alpha_i\mathbf{1}$, where $\alpha_i \in \mathbb{R}_{0+}$. Then the optimal control problem for subsystem i becomes

$$\min_{(\mathbf{v}_i, \alpha_i)} V_i(z_i, \mathbf{v}_i) + \rho\alpha_i$$

subject to

$$
\begin{aligned}
z_i(j+1) &= A_{ii}z_i(j) + B_{ii}v_i(j),\ j = 1 \ldots N-1 \\
z_i(0) &= z_i, \\
z_i(j) &\in \alpha_i\mathbb{X}_i(\mathbf{1}) \ominus \gamma_i\mathcal{R}_i(\mathbf{1}),\ j = 0 \ldots N-1 \\
v_i(j) &\in \alpha_i\mathbb{U}_i(\mathbf{1}) \ominus \gamma_i K_i\mathcal{R}_i(\mathbf{1}),\ j = 0 \ldots N-1 \\
z_i(N) &= \alpha_i\mathbb{X}_i^f(\mathbf{1},\mathbf{1};\gamma_i\mathbf{1}),
\end{aligned}
$$

Some explanations are due. In step 3a of the algorithm, subsystem controllers exchange values of the scalar α_i^*. The subsequent disturbance set for subsystem i becomes

$$\bigoplus_{j\in\mathcal{N}_i} \alpha_j^*(A_{ij}\mathbb{X}_j(\mathbf{1}) + B_{ij}\mathbb{U}_j(\mathbf{1}))$$

for which an outer approximation may be computed easily as

$$\mathbb{W}_i(g_i^+) = \gamma_i^+\mathbb{W}_i(\mathbf{1}) \supseteq \bigoplus_{j\in\mathcal{N}_i} \alpha_j^*(A_{ij}\mathbb{X}_j(\mathbf{1}) + B_{ij}\mathbb{U}_j(\mathbf{1}))$$

where $\gamma_i^+ = \max_{j\in\mathcal{N}_i}\{\alpha_j^*\}$. It follows that the RPI set using this outer-approximated disturbance set is obtained directly as

$$\mathcal{R}_i(q_i^+) = \gamma_i^+\mathcal{R}_i(\mathbf{1}).$$

Likewise, the parametric terminal set is simply scaled as shown in the above terminal constraint.

With this simplification, the sets $\mathbb{W}_i(\mathbf{1})$ and $\mathcal{R}_i(\mathbf{1})$ are computed off-line, and no Minkowski additions or ancillary LPs need to be solved on-line. The drawback is, of course, conservativeness; however, the approach is still less conservative than the conventional tube-based decentralized MPC approach, as demonstrated in Section 6.

5. Recursive feasibility and stability

One of the attractive features of tube MPC is guaranteed recursive feasibility despite the bounded disturbance: given a feasible solution $\mathbf{v}_i^*(z_i)$ to $\bar{\mathbb{P}}_i(z_i)$—where we take this to mean the conventional, non-parametric optimal control problem—with $x_i \in z_i \oplus \mathcal{R}_i$, it is simple to show that $\tilde{\mathbf{v}}_i(z_i^+)$, where $z_i^+ = A_{ii}z_i + B_{ii}\bar{\kappa}_i(z_i)$, and $\tilde{\mathbf{v}}_i(z_i^+)$ is obtained as the *tail* of the sequence $\mathbf{v}_i^*(z_i)$

$$\tilde{\mathbf{v}}_i(z_i^+) = \{v_i^*(1;z_i), v_i^*(2;z_i), \ldots, v_i^*(N-1;z_i), K_i^f z_i^*(N;z_i)\} \quad (17)$$

is feasible for $\bar{\mathbb{P}}_i(z_i^+)$. Moreover, since $x_i^+ \in z_i^+ \oplus \mathcal{R}_i$ and $\bar{\mathbb{P}}_i(z_i^+)$ includes tightened versions of $\mathbb{X}_i$ and $\mathbb{U}_i$, the true subsystem states and inputs satisfy all constraints for *any* $x_i^+ \in A_{ii}x_i + B_{ii}\kappa_i(x_i, z_i) \oplus \mathbb{W}_i$ and all future $x_i(k)$.

This is the situation when the same RPI set, $\mathcal{R}_i$, is used in the problems at x_i and x_i^+. That the tail of the previous solution is feasible at the successor state is also valuable in establishing closed-loop stability [21].

5.1. Loss of feasible tail guarantee

When a *different* RPI set is used in the problem at x_i^+, this feasible tail guarantee is destroyed.

Proposition 1 (Infeasibility of the tail). *Suppose that $\mathbf{v}_i^*(z_i)$ is feasible for $\mathbb{P}_i(z_i;q_i)$, where $\mathcal{R}_i(q_i)$ satisfies $(A_{ii}+B_{ii}K_i)\mathcal{R}_i(q_i) \oplus \mathbb{W}_i(g_i) \subseteq \mathcal{R}_i(q_i)$ for some $g_i \in \mathbb{R}_{0+}^{r_i^w}$. Consider that the RPI set is changed to $\mathcal{R}_i(q_i^+) \neq \mathcal{R}_i(q_i)$ as a result of the disturbance set changing to $\mathbb{W}_i(g_i^+) \neq \mathbb{W}_i(g_i)$. Then (i) $\tilde{\mathbf{v}}_i(z_i^+)$ is not necessarily feasible for $\mathbb{P}_i(z_i^+;q_i^+)$; (ii) the future trajectory $\{x_i(k), u_i(k)\}_k$ does not necessarily satisfy all constraints.*

Remark 3. The proof is omitted, but it is simple to construct instances of infeasibility and constraint violation, both when the RPI set is reducing and enlarging in size. For example, consider when $\mathbb{W}_i(\mathbf{0}) = \{0\}$ (which may happen when all coupled subsystems are at the origin), so that $x_i - z_i \in \mathcal{R}_i(\mathbf{0}) = \{0\}$, and the disturbance set *increases* from $\mathbb{W}_i(\mathbf{0})$ to $\mathbb{W}_i(\mathbf{1})$. Then given a feasible solution $\mathbf{v}^*(z)$ to $\mathbb{P}_i(z_i;\mathbf{0})$, $\tilde{\mathbf{v}}_i(z_i^+)$ is not necessarily feasible for $\mathbb{P}_i(z_i^+;\mathbf{1})$ because $z_i^+ \in \mathbb{X}_i \ominus \mathcal{R}_i(\mathbf{0}) = \mathbb{X}_i$ does not imply $z_i^+ \in \mathbb{X}_i \ominus \mathcal{R}_i(\mathbf{1})$. On the other hand, consider the reverse situation: when $x_i - z_i \in \mathcal{R}_i(\mathbf{1})$ and the disturbance decreases from $\mathbb{W}_i(\mathbf{1})$ to $\mathbb{W}_i(\mathbf{0}) = \{0\}$. Given a feasible solution $\mathbf{v}^*(z)$ for $\mathbb{P}_i(z_i;\mathbf{1})$, it does now follow that $\tilde{\mathbf{v}}_i(z_i^+)$ is feasible for $\mathbb{P}_i(z_i^+;\mathbf{0})$. However, $x_i^+ - z_i^+$ is not necessarily in $\mathcal{R}_i(\mathbf{0}) = \{0\}$, meaning that constraint satisfaction by the *true* subsystem dynamics is not guaranteed for all $\mathbf{v}_i \in \mathcal{V}_i(z_i^+)$.

Proposition 1 has profound implications. If it cannot be guaranteed that a feasible solution can be constructed from the tail of a previous one, then the tail can not be used in the usual way to establish monotonic descent of the value function and, hence, stability of the system. It is this that motivates the checking step in the algorithm, and in the next section we show that, with this step included and an additional assumption, the tail feasibility guarantee is maintained.

For the remainder of Section 5, the standing assumptions 1–4 are supposed to hold.

5.2. Non-increasing disturbance sets imply feasibility

We begin with establishing a sufficient condition for guaranteed feasibility of the tail. For this, we require the following two lemmas.

Lemma 2 (Smaller $\mathbb{W}_i$ implies smaller $\mathcal{R}_i$). *Suppose that $\mathcal{R}_i(q_i^1)$ satisfies $(A_{ii}+B_{ii}K_i)\mathcal{R}_i(q_i^1) \oplus \mathbb{W}_i(g_i^1) \subseteq \mathcal{R}_i(q_i^1)$ for some $g_i^1 \in \mathbb{R}_{0+}^{r_i^w}$. Consider $g_i^2 \leq g_i^1$, so $\mathbb{W}_i(g_i^2) \subseteq \mathbb{W}_i(g_i^1)$. Then there exists $q_i^2 \leq q_i^1$ such that $\mathcal{R}_i(q_i^2) \subseteq \mathcal{R}_i(q_i^1)$ and $(A_{ii}+B_{ii}K_i)\mathcal{R}_i(q_i^2) \oplus \mathbb{W}_i(g_i^2) \subseteq \mathcal{R}_i(q_i^2)$.*

Proof. If $(A_{ii}+B_{ii}K_i)\mathcal{R}_i(q_i^1) \oplus \mathbb{W}_i(g_i^1) \subseteq \mathcal{R}_i(q_i^1)$ then $(A_{ii}+B_{ii}K_i)\mathcal{R}_i(q_i^1) \oplus \mathcal{W} \subseteq \mathcal{R}_i(q_i^1)$ for any $\mathcal{W} \subseteq \mathbb{W}_i(g_i^1)$, including $\mathbb{W}_i(g_i^2)$. Therefore, $q_i^2 = q_i^1$ is a valid choice to satisfy the claim. □

Lemma 3 (Smaller $\mathcal{R}_i$ implies larger $\mathcal{Z}_i$). *Given q_i^1, q_i^2 such that $q_i^2 \leq q_i^1 \leq \mathbf{1}$, $\mathcal{Z}_i(q_i^1) \subseteq \mathcal{Z}_i(q_i^2)$.*

Proof. Given some $z_i \in \mathcal{Z}_i(q_i^1)$, by definition there exists a $\mathbf{v}_i^*(z_i) \in \mathcal{V}_i(a_i^*, b_i^*; q_i^1)$ where $a_i^* \leq \mathbf{1}$ and $b_i^* \leq \mathbf{1}$. The same $\mathbf{v}_i^*(z_i) \in \mathcal{V}_i(a_i^*, b_i^*; q_i^2)$, for $q_i^2 \leq q_i^1$, in view of the constraints set inclusion $\mathbb{X}_i(a_i^*) \ominus \mathcal{R}_i(q_i^1) \subseteq \mathbb{X}_i(a_i^*) \ominus \mathcal{R}_i(q_i^2)$, with similar inclusions for $\mathbb{U}_i$ and $\mathbb{X}_i^f$. Therefore, $z_i \in \mathcal{Z}_i(q_i^2)$. □

Proposition 2 (Condition for tail feasibility). *Suppose $z_i \in \mathcal{Z}_i(q_i)$ and $(\mathbf{v}_i^*(z_i), a_i^*, b_i^*)$ is a feasible solution to $\mathbb{P}_i(z_i; q_i)$. Then $(\tilde{\mathbf{v}}_i(z_i^+), a_i^*, b_i^*)$ is a feasible solution to $\mathbb{P}_i(z_i^+; q_i^+)$ if $q_i^+ \leq q_i$.*

Proof. Follows directly from Lemma 3 and the definition of $\mathcal{Z}_i$. □

This sufficient condition, together with Lemmas 2 and 3, motivates the following assumption, which we shall use to establish recursive feasibility. A discussion on the strength, implications and satisfaction of this assumption is given in Section 5.3.

Assumption 5. The optimized set parameters a_i and b_i are non-increasing over time: *i.e.*, $a_i^*(k) \leq a_i^*(k-1)$ and $b_i^*(k) \leq b_i^*(k-1)$.

The following result establishes recursive feasibility of the controlled system.

Theorem 1 (Recursive feasibility). *Suppose that Assumption 5 holds and, for all $i \in \mathcal{N}$, that $x_i(0) \in \mathcal{Z}_i(\mathbf{1})$. Then each subsystem controlled according to Algorithm 1 is recursively feasible and satisfies all constraints.*

Proof. Consider subsystem $i \in \mathcal{N}$ with state (x_i, z_i) and suppose there exists a solution $(\mathbf{v}_i^*(z_i), a_i^*, b_i^*)$ to $\mathbb{P}_i(z_i, q_i)$. Further suppose that $x_i - z_i \in \mathcal{R}_i(q_i)$, so that $x_i \in \mathbb{X}_i$, $u_i \in \mathbb{U}_i$. Now consider the successor sets $\mathbb{W}_i(g_i^+) \subseteq \mathbb{W}_i(g_i)$ and $\mathcal{R}_i(q_i^+) \subseteq \mathcal{R}_i(q_i)$, where, because of Assumption 5, $g_i^+ \leq g_i$ and $q_i^+ \leq q_i$. The successor error state is $(x_i^+ - z_i^+) \in (A_{ii} + B_{ii}K_i)(x_i - z_i) \oplus \mathbb{W}_i(g_i^+)$: by construction, $(x_i^+ - z_i^+) \in \mathcal{R}(q_i)$, but either (i) $(x_i^+ - z_i^+) \in \mathcal{R}(q_i^+)$ or (ii) $(x_i^+ - z_i^+) \notin \mathcal{R}(q_i^+)$. In either case, by Proposition 2, $(\tilde{\mathbf{v}}_i(z_i^+), a_i^*, b_i^*)$ is a feasible solution to both $\mathbb{P}_i(z_i^+, q_i^+)$ and $\mathbb{P}_i(z_i^+, q_i)$. In case (i), q_i^+ is adopted, and the future trajectory of the system satisfies $x_i(k) \in z_i(k) \oplus \mathcal{R}_i(q_i^+)$ for any sequence $\{w_i(k)\} \in \mathbb{W}_i(g_i^+) \times \mathbb{W}_i(g_i^+) \times \ldots$. Moreover, $x_i(k) \in \mathbb{X}_i$, $u_i(k) \in \mathbb{U}_i$ because $z_i(k) \in \mathbb{X}_i \ominus \mathcal{R}_i(q_i^+)$, $v_i(k) \in \mathbb{U}_i \ominus K_i\mathcal{R}_i(q_i^+)$. In case (ii), q_i^+ is not adopted, keeping q_i, and the future trajectory of the system satisfies $x_i(k) \in z_i(k) \oplus \mathcal{R}_i(q_i)$ also for any sequence $\{w_i(k)\} \in \mathbb{W}_i(g_i^+) \times \mathbb{W}_i(g_i^+) \times \ldots$. Moreover, $x_i(k) \in \mathbb{X}_i$, $u_i(k) \in \mathbb{U}_i$ because $z_i(k) \in \mathbb{X}_i \ominus \mathcal{R}_i(q_i)$, $v_i(k) \in \mathbb{U}_i \ominus K_i\mathcal{R}_i(q_i)$. This completes the part of the proof that, given a feasible solution with $x_i - z_i \in \mathcal{R}_i(q_i)$, feasibility and constraint satisfaction are guaranteed for all subsequent times.

Now consider time 0. $x_i(0) \in \mathcal{Z}_i(\mathbf{1})$ so by definition there exists a solution $(\mathbf{v}_i^*(z_i), a_i^*, b_i^*)$ to $\mathbb{P}_i(z_i(0), q_i(0))$ where $z_i(0) = x_i(0)$ and $q_i(0) = \mathbf{1}$. It follows that, for all $i \in \mathcal{N}$, $\mathbb{W}_i(g_i(1)) = \mathbb{W}_i(g_i^+) \subseteq \mathbb{W}_i(g_i(0)) = \mathbb{W}_i(\mathbf{1})$, hence $\mathcal{R}_i(q_i(1)) = \mathcal{R}_i(q_i^+) \subseteq \mathcal{R}_i(q_i) = \mathcal{R}_i(\mathbf{1})$. The successor error state is $x_i(1) - z_i(1) \in (A_{ii} + B_{ii}K_i)(x_i(0) - z_i(0)) \oplus \mathbb{W}_i(g_i(0)) = \mathbb{W}_i(g_i(0)) \subseteq \mathcal{R}_i(q_i(1))$, so (i) $x_i(1) \in \mathbb{X}_i$ for all $w_i(0) \in \mathbb{W}_i(g_i(0))$ and (ii) the check in step 5 is satisfied. So, $q_i(1) \leq q_i(0) = \mathbf{1}$ is adopted for time 1, and, using the preceding part of this proof for time $k = 1$ onwards, the result is established. □

The final result of this section establishes exponential stability of the origin for the controlled system. The following assumption is made.

Assumption 6. The gain matrix $K \triangleq \mathrm{diag}(K_1, K_2, \ldots, K_M)$ is such that the large-scale system $x^+ = (A + BK)x$ has all eigenvalues strictly within the unit circle.

Remark 4 (Mildness of Assumption 6). In theory, determining suitable K_i such that all $(A_{ii} + B_{ii}K_i)$, and also $(A + BK)$, are stable is a non-trivial problem. It may be cast conservatively as a linear matrix inequality (LMI) problem of designing a static state feedback controller $u = Kx$ for $x^+ = Ax + Bu$ with decentralized structure imposed on K [33, 34]. In practice, however, it is desirable to design the K_i such that the sets $\mathcal{R}_i$ are small, which suggests the poles of $(A_{ii} + B_{ii}K_i)$ being close to, or at, the origin. In that case, the coupling between subsystems would need to be relatively strong (relatively large off-diagonal A_{ij} and B_{ij} compared to A_{ii} and B_{ii}) in order for the eigenvalues of $(A + BK)$ to lie outside of the unit circle.

Theorem 2 (Exponential stability of the origin). *Suppose Assumptions 5 and 6 hold. Then the origin is exponentially stable for each subsystem $i \in \mathcal{N}$ when controlled according to Algorithm 1. The region of attraction is $\mathcal{Z}_i(\mathbf{1})$.*

Proof. We first show exponential stability of the origin for the nominal controlled subsystem, $z_i^+ = A_{ii}z_i + B_{ii}\bar{\kappa}_i(z_i)$, and then use the fact that $x_i \in z_i \oplus \mathcal{R}_i(q_i)$, together with the stable large-scale dynamics, to show the same for the true state.

Given some $z_i \in \mathcal{Z}_i(\mathbf{1})$, the optimal cost of problem $\mathbb{P}_i(z_i; q_i)$ is $V_i(z_i, \mathbf{v}_i^*(z_i)) + \rho_a\|a_i^*\|_1 + \rho_b\|b_i^*\|_1$. The value function $V_i^*(z_i; q_i) = V_i(z_i, \mathbf{v}_i^*(z_i))$ satisfies

$$c_i|z_i|^2 \leq V_i^*(z_i; q_i) \leq d_i|z_i|^2, \forall z_i \in \mathcal{Z}_i(q_i), \mathbf{0} \leq q_i \leq \mathbf{1}$$

where $d_i > c_i > 0$. The lower bound here follows by definition of $V_i(z_i, \mathbf{v}_i)$, and the upper bound from continuity of $V_i^*(z_i; q_i)$, which itself follows from the fact that $z_i^+ = A_{ii}z_i + B_{ii}v_i$ is linear and the sets $\mathbb{X}_i(a_i) \ominus \mathcal{R}_i(q_i)$, $\mathbb{U}_i(b_i) \ominus K_i\mathcal{R}_i(q_i)$, and $\mathbb{X}_i^f(a_i, b_i; q_i)$ are polytopic. Under Assumption 5, an upper bound on the optimal cost of $\mathbb{P}_i(z_i^+; q_i^+)$ is $V_i(z_i^+, \tilde{\mathbf{v}}_i(z_i^+)) + \rho_a\|a_i^*\|_1 + \rho_b\|b_i^*\|_1$ where

$$V_i(z_i^+, \tilde{\mathbf{v}}_i(z_i^+)) \leq V_i^*(z_i; q_i) - \ell_i(z_i, \bar{\kappa}_i(z_i)).$$

Moreover, since, by Assumption 5, $q_i^+ \leq q_i$, then $V_i^*(z_i^+; q_i^+) \leq V_i^*(z_i^+; q_i) \leq V_i(z_i^+, \tilde{\mathbf{v}}_i(z_i^+))$. Hence

$$V_i^*(z_i^+; q_i^+) - V_i^*(z_i; q_i) \leq -\ell_i(z_i, \bar{\kappa}_i(z_i)) \leq -c_i|z_i|^2,$$

for all $z \in \mathcal{Z}_i(q_i), \mathbf{0} \leq q_i \leq \mathbf{1}, q_i^+ \leq q_i$. By the usual arguments (see, for example, [21, Theorem 2.24]), there exists a constant $\gamma_i > 0$ such that $|z_i(k)| \leq \gamma_i(1 - c_i/d_i)^k|z_i(0)|$, and the origin is therefore exponentially stable for $z_i^+ = A_{ii}z_i + B_{ii}\bar{\kappa}_i(z_i)$ with region of attraction $\mathcal{Z}_i(\mathbf{1})$.

Since $z_i(0) = x_i(0)$, then $x_i(k) \in z_i(k) \oplus \mathcal{R}_i(\mathbf{1})$ for all k. More specifically, $x_i(k) \in z_i(k) \oplus \mathcal{R}_i(q_i(k))$ at some k, where $\mathbf{0} \leq q_i(k) \leq \mathbf{1}$. Since $x_i(k) = z_i(k) + e_i(k)$, where $e_i(k) \triangleq x_i(k) - z_i(k) \in \mathcal{R}_i(q_i(k))$, then

$$|x_i(k)|_{\mathcal{R}_i(q_i(k))} = |z_i(k) + e_i(k)|_{\mathcal{R}_i(q_i(k))} \leq |z_i(k) + e_i(k)|_{e_i(k)} = |z_i(k)|$$

and so

$$|x_i(k)|_{\mathcal{R}_i(q_i(k))} \leq \gamma_i(1 - c_i/d_i)^k |z_i(0)|, \tag{18}$$

which shows the distance between $x_i(k)$ and $\mathcal{R}_i(q_i(k))$ decreases exponentially fast. $\mathcal{R}_i$ varies with $q_i(k)$, and we wish to establish convergence of the sequence $\{q_i(k)\}$ to a limit; first, consider the sequences $\{a_i^*(k)\}$, $\{b_i^*(k)\}$, where $\mathbf{0} \leq a_i^* \leq \mathbf{1}$ and $a_i^*(k) \leq a_i^*(k-1)$ for all k by Assumption 5, with similar bounds for b_i^*. Thus, each is a non-increasing, bounded sequence that converges to some finite limit, say $(\bar{a}_i, \bar{b}_i)$. It holds that $(\mathbf{0}, \mathbf{0}) \leq (\bar{a}_i, \bar{b}_i) \leq (\mathbf{1}, \mathbf{1})$. Then $\mathbb{W}_i(g_i) \to \mathbb{W}_i(\bar{g}_i) = \bigoplus_{j \in \mathcal{N}_i} (A_{ij}\mathbb{X}_j(\bar{a}_i) \oplus B_{ij}\mathbb{U}_j(\bar{b}_j))$, and $\mathcal{R}_i(q_i) \to \mathcal{R}_i(\bar{q}_i)$. Because of (18), the state x_i converges to $\mathcal{R}_i(\bar{q}_i)$ exponentially fast.

As a consequence of the exponential convergence of the nominal state z_i, there exists a k^* such that every nominal state z_i enters a set $\{0\} \oplus \mathcal{B}(\delta)$ in finite time k^*; the true state x_i lies within $\mathcal{B}(\delta) \oplus \mathcal{R}_i(q_i(k^*))$. Within this set, the large-scale system dynamics evolve according to

$$x^+ = Ax + Bu = (A + BK)x + B(v^* - Kz)$$

where $v_i^* = \bar{\kappa}_i(z_i)$; moreover, there exists a sufficiently small δ such that, for $z_i \in \{0\} \oplus \mathcal{B}(\delta)$, the optimal sequences $\{v_i^*(0), \ldots, v_i^*(N-1)\}$ and $\{z_i^*(0), \ldots, z_i^*(N)\}$ lie in the interiors of their respective constraint sets, and the control law $\bar{\kappa}_i(z_i) = \bar{K}_i z_i$. Then $v^* - Kz = (\bar{K} - K)z$ where $\bar{K} = \text{diag}(\bar{K}_1, \ldots, \bar{K}_M)$. Owing to the exponential convergence of each z_i, the term $B(v^* - Kz) = B(\bar{K} - K)z$ is exponentially decaying. In view of this and stability of $(A + BK)$, we conclude that the state x, and hence each subsystem state x_i, in fact converges exponentially to the origin. □

5.3. Discussion: ensuring non-increasing disturbance sets

The recursive feasibility guarantee relies on Assumption 5, and so naturally the question arises of how strong it is and when it is met. The analysis in the previous section informs about the sufficiency of non-increasing (a_i^*, b_i^*) in order to guarantee recursive feasibility and constraint satisfaction: for a subsystem i at time k, if (a_j^*, b_j^*) of neighbours $j \in \mathcal{N}_i$ are non-increasing with respect to their previous values, then the RPI set $\mathcal{R}_i(q_i^+)$ does not increase in size and (i) there is guaranteed to exist a solution to the modified optimal control problem at the next step $k+1$—namely, the candidate solution $\tilde{\mathbf{v}}_i(z_i^+)$ together with set parameters $a_i^*(k+1) = a_i^*(k)$ and $b_i^*(k+1) = b_i^*(k)$—and (ii) all constraints are guaranteed to be satisfied. By recursion and extension across the whole system, therefore, Assumption 5 is automatically met if each subsystem were to adopt the (suboptimal) feasible candidate solution at every step.

For performance reasons, however, it is desirable to obtain an optimal solution at each time step. Yet it does not hold, in general, that non-increasing values of a_i and b_i are an optimal choice: the optimal a_i and b_i depend on the choice of weighting parameters in the cost function (*i.e.*, the Q_i, R_i, ρ_a and ρ_b), and the application, and it is possible that the objectives of minimizing $V_i(z_i, \mathbf{u}_i)$ and $\rho_a \|a_i\|_1 + \rho_b \|b_i\|_1$ are conflicting. On the one hand, the controller would like to optimize predicted control performance, which it is most free to do so when the its constraint sets are very large, but on the other hand it desires to minimize the size of the disturbance sets, which lowers conservativeness across the system. Therefore, a balance must be reached between the size of the constraints sets and the maximal size of the disturbance sets: ideally, very small penalties on a_i and b_i will incentivize the controllers to eliminate the slack or excess in the constraint sets without adversely affecting the optimal state and input trajectories.

In the event that increasing a_i and b_i do occur, subsequent feasibility and constraint satisfaction may be lost (but not necessarily so, since the non-increase of (a_i, b_i) is merely a sufficient condition, rather than a necessary one, for recursive feasibility). However, we also note that the option always exists to reject such a solution and adopt the feasible candidate solution, which is formed from a previously computed solution and satisfies Assumption 5.

In the following sections, we give some further guidelines and considerations for design in order to maintain feasibility and good performance.

5.3.1. Constraining non-increase of (a_i, b_i)

Satisfaction of Assumption 5 is guaranteed if non-increase of (a_i, b_i) is constrained, either as a constraint in the optimization, *i.e.*,

$$\begin{aligned} a_i &\leq a_i^*(k-1), \\ b_i &\leq b_i^*(k-1), \end{aligned}$$

or as an extra step and condition within the algorithm; that is, an optimal solution is adopted if and only if $a_i^* \leq a_i^*(k-1)$ and $b_i \leq b_i^*(k-1)$. The former has the disadvantage that the optimization problems become increasingly constrained, and the controllers have less flexibility and robustness, as the subsystem states converge; an external disturbance, unmodelled uncertainty or setpoint change could easily render the system infeasible. The latter option appears to avoid this, but in fact the controllers still lose flexibility and robustness since only solutions that lead to non-increase of a_i and b_i can be implemented.

5.3.2. Promoting non-increase via the objective

A preferable option, which retains flexibility within the controllers, is to promote non-increase via the cost function, either by penalizing more heavily (via ρ_a and ρ_b) the values of a_i and b_i, or by explicitly penalizing increase of a_i and b_i. For the latter, non-increase can be promoted via the objective

$$V_i(z_i, \mathbf{u}_i) + \rho_a \|a_i\|_1 + \rho_b \|b_i\|_1 + W(\gamma_i^a + \gamma_i^b),$$

where γ_i^a, γ_i^b are non-negative scalar variables, W is large and positive, and

$$\begin{aligned} a_i &\leq a_i^*(k-1) + \gamma_i^a \\ b_i &\leq b_i^*(k-1) + \gamma_i^b. \end{aligned}$$

In this case, the controller is permitted to select a solution with increasing (a_i, b_i) if it needs to, but prefers not to.

The precise selection and tuning of the weights Q_i, R_i, ρ_a, ρ_b and W depends on the particular application. However, the price of meeting Assumption 5 in this way is suboptimality with respect to the control objective. This, and the relaxation of Assumption 5, are topics of current research.

6. Illustrative example

We consider a modified version of the four-truck system from [15]. Each truck is modelled by the second-order dynamics

$$\begin{bmatrix}\dot{r}_i\\ \dot{v}_i\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1\\ -\frac{1}{m_i}\sum_{j\in\mathcal{N}_i}k_{ij} & -\frac{1}{m_i}\sum_{j\in\mathcal{N}_i}h_{ij}\end{bmatrix}}_{A_{ii}}\begin{bmatrix}r_i\\ v_i\end{bmatrix} + \begin{bmatrix}0\\ 100\end{bmatrix}u_i + w_i$$

$$w_i = \sum_{j\in\mathcal{N}_i}\underbrace{\begin{bmatrix}0 & 0\\ \frac{1}{m_i}\sum_{j\in\mathcal{N}_i}k_{ij} & \frac{1}{m_i}\sum_{j\in\mathcal{N}_i}h_{ij}\end{bmatrix}}_{A_{ij}}\begin{bmatrix}r_j\\ v_j\end{bmatrix}$$

where r_i is the displacement of truck i from an equilibrium position, v_i is its velocity and u_i is the control input (acceleration). The disturbance w_i arises via the coupling between trucks: truck 1 (mass $m_1 = 3\,\text{kg}$) is coupled to truck 2 (mass $m_2 = 2\,\text{kg}$) via a spring (stiffness $k_{12} = 0.5\,\text{N}\,\text{m}^{-1}$) and damper ($h_{12} = 0.2\,\text{N}\,\text{m}^{-1}\,\text{s}^{-1}$). Likewise, truck 3 (mass $m_3 = 3\,\text{kg}$) is coupled to truck 4 (mass $m_4 = 6\,\text{kg}$) via $k_{34} = 1\,\text{N}\,\text{m}^{-1}$ and $h_{34} = 0.3\,\text{N}\,\text{m}^{-1}\,\text{s}^{-1}$. However, in this paper we modify the system to also couple trucks 2 and 3 via $k_{23} = 0.75\,\text{N}\,\text{m}^{-1}$ and $h_{23} = 0.25\,\text{N}\,\text{m}^{-1}\,\text{s}^{-1}$, so that the four trucks are coupled as one group.

The problem considered is controlling the trucks to equilibrium from initial states of

$$x_1 = \begin{bmatrix}1.8\\ 0\end{bmatrix} \quad x_2 = \begin{bmatrix}-0.5\\ 0\end{bmatrix} \quad x_3 = \begin{bmatrix}1\\ 0\end{bmatrix} \quad x_4 = \begin{bmatrix}-1\\ 0\end{bmatrix}.$$

The trucks are subject to state constraints $|r_i| \leq 4$, $|v_i| \leq 1$ and input constraints $|u_i| \leq 1$ for $i = 1, 2, 3$, $|u_4| \leq 2$. These constraints form the sets $\mathbb{X}_i(\mathbf{1})$, $\mathbb{U}_i(\mathbf{1})$, from which the initial disturbance sets $\mathbb{W}_i(\mathbf{1})$ are computed.

For the DMPC design, the continuous-time dynamics are discretized using zero-order hold and a sampling time of 0.1 seconds, treating the state couplings as exogenous disturbances in order to preserve sparsity of the subsystem-to-subsystem coupling. (Note, then, that the discretization is approximate rather than exact—for an interesting discussion and contribution toward sparsity-preserving discretization, see [35].) The MPC controllers are designed with cost matrices $Q_i = I$ and $R_i = 100$. The weighting parameter ρ_a—which governs the preference for minimizing the system cost versus minimizing the size of state constraint set—was set to 0.0001. Because the trucks are not input coupled, we fix $\mathbb{U}_i = \mathbb{U}_i(\mathbf{1})$ and do not include b_i as an optimization variable. The prediction horizon is $N = 25$.

For each truck, the tube control law, K_i, is chosen to be the deadbeat controller for the local nominal dynamics $x_i^+ = A_{ii}x_i + B_iu_i$. This means the minimal RPI set is finitely determined, and leads to an initial RPI set, $\mathcal{R}_i(\mathbf{1})$, defined by four inequalities. On the other hand, the parametric terminal set is the maximal constraint admissible set associated with the infinite-horizon LQR terminal controller $K_i^f = K_\infty(A_{ii}, B_i, Q_i, R_i)$ and the state and input constraint sets $\mathbb{X}_i(a_i)$ and $\mathbb{U}$; the terminal cost matrix P_i is the corresponding Lyapunov equation solution.

6.1. Constraint and invariant set comparison

Figure 1 shows the relevant constraint, terminal and tube sets for trucks 1 and 2. The top subfigure illustrates the potential drawback of taking a robust approach to what is a nominal control problem, for the tube set $\mathcal{R}_2(\mathbf{1})$ for truck 2—constructed to offer robustness to the disturbance set $\mathbb{W}_2(\mathbf{1})$ induced by the couplings with trucks 1 and 3—is of a significant size compared to the state constraint set $\mathbb{X}_2(\mathbf{1})$. The tightening of the state constraint set—the difference between the set $\mathbb{X}_2(\mathbf{1})$ and the tightened set $\mathbb{X}_2(\mathbf{1}) \ominus \mathcal{R}_2(\mathbf{1})$—is significant.

The middle subfigure of Figure 1 shows the same sets for truck 1, but also the re-configured versions of these sets after having solved the initial optimal control problem at time $k = 0$. The state constraint set $\mathbb{X}_1(\mathbf{1})$ is optimized to $\mathbb{X}_1(a_1^*)$, while the parametric terminal set $\mathbb{X}_1^f(\mathbf{1};\mathbf{1})$ becomes $\mathbb{X}_1^f(a_1^*;\mathbf{1})$. Note the asymmetry of the re-configured state and terminal sets, and that the latter is, of course, a subset of the former. The optimized state trajectory $\mathbf{x}_1^*$ is also shown, and respects all constraints. Finally, the bottom subfigure illustrates the impact of this minimization of the constraint set for truck 1 on the tube cross-section (RPI) set of the coupled truck 2, which is reduced significantly.

6.2. Closed-loop performance

To evaluate the performance of the proposed scheme, the four-truck system is controlled by three different algorithms:

1. Algorithm 1 (“DMPC”).
2. Tube Decentralized MPC (“Tube DeMPC)”, which is Algorithm 1 without optimization of a_i and re-computation of RPI sets.
3. Centralized MPC (“CMPC”).

Figure 2 shows the resulting closed-loop state trajectories. For Tube DeMPC and the proposed DMPC, the RPI sets are also shown at each time step. Note that these represent uncertainty around nominal trajectories, and each controller has to tighten constraints to make allowance for the entire tube cross-section. For Tube DeMPC these sets are of a considerable size, meaning the velocity constraint must be tightened significantly. In contrast, the RPI sets for Alg. 1 contract significantly after the initial step, as controllers form better estimates of the actual disturbance sets, based on how much of the original state and input sets each subsystem is using. The bottom plot of Figure 2, showing the first five steps of truck 1, illustrates this contraction clearly. With Algorithm 1, the RPI sets are significantly reduced after just one time step, and the subsystem trajectory is able to go closer to constraints and the CMPC trajectory.

Table 1 compares the total closed-loop costs obtained from the different approaches, showing results for using as a terminal set both the reconfigurable maximal admissible set, $\mathbb{X}_i^f = \mathcal{O}_i^\infty(a_i, b_i; q_i)$, and the origin. As the state trajectories shown in the figures suggest, the proposed approach achieves performance closer to that of centralized MPC than that of Tube DeMPC. The simplified implementation of the proposed DMPC (“sDMPC”),

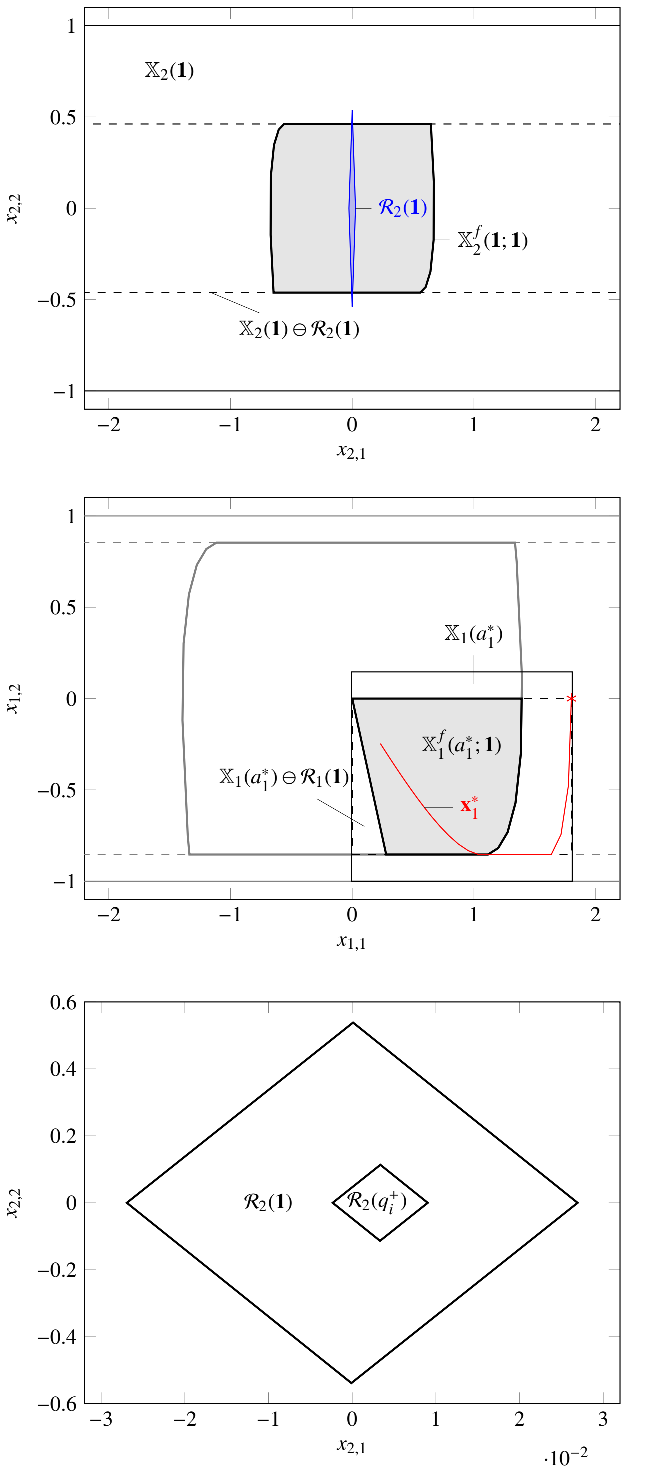

Figure 1: (Top) State constraint set $\mathbb{X}_2(\mathbf{1})$, terminal set $\mathbb{X}_2^f(\mathbf{1};\mathbf{1})$, RPI set $\mathcal{R}_2(\mathbf{1})$, and tightened state constraint set $\mathbb{X}_2(\mathbf{1}) \ominus \mathcal{R}_2(\mathbf{1})$ for truck 2. (Middle) The same sets for truck 1 both prior to (lighter lines), and following (darker lines), the optimization at time $k = 0$; $\mathbb{X}_1(\mathbf{1})$ is reduced to $\mathbb{X}_1(a_1^*)$ and the parametric terminal set is reduced from $\mathbb{X}_1^f(\mathbf{1};\mathbf{1})$ to $\mathbb{X}_1^f(a_1^*;\mathbf{1})$. Also shown is the optimized state trajectory $\mathbf{x}_1^*$. (Bottom) The original RPI set $\mathcal{R}_2(\mathbf{1})$ for truck 2, and the new set, $\mathcal{R}_2(q_2^+)$, computed after receiving a_1^* from truck 1.

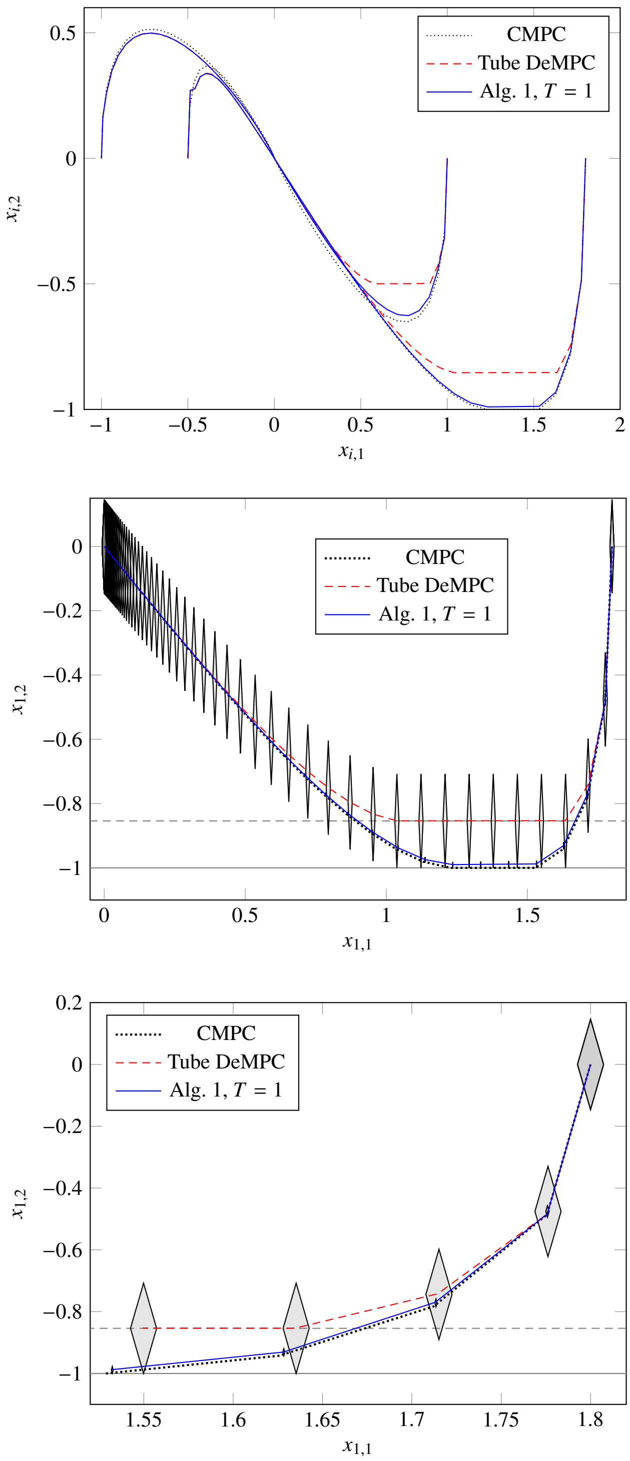

Figure 2: Closed-loop state trajectories of system controlled by CMPC, Tube DeMPC, and Alg. 1 with $T = 1$. (Top) All four trucks. (Middle) Truck 1, with tube cross-section sets shown. (Bottom) Enlarged image of the first five steps of truck 1's trajectory.

described in Section 4.3, is also included in the table; it can be seen to out-perform the DeMPC control scheme, despite the minimal on-line computational complexity. Finally, the effect of the update rate T of the RPI sets is shown; good performance is achieved even with $T = 10$.

6.3. On-line computation

Regarding the computational complexity of the approach, solving the distributed optimal control problem (which is a QP) took a maximum time 0.007 seconds across the four trucks during the 100-step simulation, using CPLEX 12.6 on a 64-bit Intel Core i7-2600 machine running at 3.40 GHz with 8 GB RAM. In contrast, the unmodified distributed optimal control problem employed by the DeMPC took a maximum time of 0.006 seconds, while the larger QP used by the centralized controller took a maximum time of 0.045 s to solve.

The ancillary LP—used to compute the RPI sets from the shared constraint set information—took a maximum time of 0.003 seconds, using CPLEX 12.6 as the LP solver, indicating the practicality of the proposed approach.

7. Conclusions

A novel tube-based DMPC scheme has been proposed with guaranteed recursive feasibility and stability. The rationale of the approach lies in the optimization of—and exchange of information about—the input and state constraint sets in order to minimize the mutual disturbances between subsystems. In order to guarantee feasibility and stability, the approach employs a parametric terminal constraint set, which adjusts automatically to account for the changes to state and input constraint sets. The re-configuration of disturbance sets and tube (RPI) sets, in response to new information from neighbours, is done on-line via the solving of a single LP by each subsystem controller. As it is verified in the simulation section, the proposed approach is less conservative than conventional tube-based decentralized MPC.

Table 1: Closed-loop costs, $\sum_k \sum_i \ell_i(x_i(k), u_i(k))$.

$\mathbb{X}_i^f$	CMPC	DeMPC	DMPC		sDMPC	
			$T = 1$	$T = 10$	$T = 1$	$T = 10$
$O_i^\infty(a_i, b_i; q_i)$	36.5826	37.2003	36.6877	36.6993	37.0247	37.0285
$\{0\}$	36.6167	37.2287	36.7023	36.7145	37.0578	37.0669

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