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paper is to demonstrate SR in continuous and spiking neuron models using arbitrary driving Lévy processes. A Lévy process is essentially a stochastic process with stationary and independent increments. Examples are Brownian motion, the Poisson process, and also non-Gaussian α-stable processes (0 < α < 2) which have infinite variance (and also infinite mean if α ≤ 1) and self-similar sample paths. In [5], Patel and Kosko show that Lévy noise can lead to SR in noisy feedback neuron models where the noise enters additively, but they required the assumption that the Lévy noise has a finite second moment. This excludes many important examples, such as the α-stable processes mentioned above, where simulation indicates that SR will also occur. The purpose of this brief is to show that the finite second moment assumption can be dropped and so to establish SR for arbitrary driving Lévy processes.

We use the same notation and setup as in [5] so our driving noise is a Lévy process \( L_t = (L_t^1, L_t^2, \ldots, L_t^m) \) taking values in \( \mathbb{R}^m \) that is defined on a probability space \((\Omega, \mathcal{F}, P)\) which is equipped with a filtration \( (\mathcal{F}_t, t \geq 0) \). As in [5], we make the convenient assumption that each \( L_t^j (1 \leq j \leq m) \) is a 1-D Lévy process and that these component processes are independent. We employ the Lévy–Itô decomposition (see, e.g., [1]) to decompose the component process \( L_t^j \) into continuous and jump parts

\[
L_t^j = \mu^j t + \sigma^j B_t^j + \int_{|y| < 1} y^j \tilde{N}^j(t, dy^j) + \int_{|y| \geq 1} y^j N^j(t, dy^j)
\]  

(1)

where for each \( 1 \leq j \leq m \), \( \mu^j \in \mathbb{R} \) \( \sigma^j \geq 0 \), \( (B_t^j, t \geq 0) \) is a standard Brownian motion \( (B_t, t \geq 0) \) which is independent of the Brownian motion \( B_t \) and has intensity measure \( d\mu^j(dy^j) \) where \( \nu^j \) is a Lévy measure. The compensated random measure is \( \tilde{N}^j(t, dy^j) = N^j(t, dy^j) - \int_0^t \sigma^j(B_s^j) \, ds \) for each \( 1 \leq j \leq m \). Define \( P_t^j = \int_{|y| \geq 1} y^j N^j(t, dy^j) \) and \( M_t^j = L_t^j - P_t^j \). Then, \((M_t^j, t \geq 0)\) and \((P_t^j, t \geq 0)\) are independent Lévy processes where the jump sizes of the process \( M_t^j \) are all bounded by one. It follows from [1, Th. 2.4.7] that \( M_t^j \) has finite moments to all orders.

To describe continuous neuron models with additive Lévy noise, Patel and Kosko [5] introduce the stochastic differential equation (SDE)

\[
dx_t = b(x_{t-}) \, dt + c(x_{t-}) \, dL_t
\]  

(2)

where \( x_t = (x_t^1, \ldots, x_t^m) \), \( b^i \) and \( c_j^i \) are globally Lipschitz functions, and we have the global bound

\[
\sup_{x \in \mathbb{R}} |c_j^i(x)|^2 \leq H_j^i.
\]  

(3)

In order to focus on “pure noise” effects, we take \( \mu^j = 0 \) as in [5]. There is no loss of generality here as \( \mu^j \) can always be incorporated into the drift term \( b \). Now consider the noiseless version of (2)

\[
dx_t = b(\hat{x}_{t-}) \, dt.
\]  

(4)

A key step on the way to obtaining SR in [5] is Lemma 1 therein which states that the solution to (2) converges to that of (4) in probability as the noise dissipates to zero. Specifically, it is shown that (under the square-integrability assumption) for all \( T > 0, K > 0 \)

\[
P \left( \sup_{0 \leq t \leq T} \|X_t - \hat{X}_t\| > K \right) \rightarrow 0
\]  

(5)

as \( \sigma^j \rightarrow 0 \) and \( \nu^j \rightarrow 0 \) for all \( 1 \leq j \leq m \). The remainder of this brief is concerned with the extension of (5) to general Lévy noise. Specifically, we have the following.

**Theorem 1:** For each \( 1 \leq j \leq d \), let \( L_t^j \) be an arbitrary real-valued Lévy process [so it has the form (1)] and assume that the \( L_t^j \)’s are independent stochastic processes. Then, for all \( T > 0, K > 0 \)

\[
P \left( \sup_{0 \leq t \leq T} \|X_t - \hat{X}_t\| > K \right) \rightarrow 0
\]  

as \( \sigma^j \rightarrow 0 \) and \( \nu^j \rightarrow 0 \) for all \( 1 \leq j \leq m \).

**Proof:** We first rewrite (2) as

\[
dx_t = b(x_{t-}) \, dt + c(x_{t-}) \, dM_t + c(x_{t-}) \, dP_t.
\]  

(6)

For each \( 1 \leq i \leq d \) and \( t \geq 0 \), define

\[
Z_i(t) = \int_0^t c_i(x_{s-}) \, dP_s^j = \int_0^t \int_{|y| \geq 1} c_i^j(x_{s-}) y^j N^j(s, dy^j)
\]

and write \( Z(t) = (Z_1(t), \ldots, Z_d(t)) \). By (5), we have

\[
P \left( \sup_{0 \leq t \leq T} \|X_t - \hat{X}_t - Z(t)\| > K \right) \rightarrow 0
\]  

(7)

as \( \sigma^j \rightarrow 0 \) and \( \nu^j \rightarrow 0 \) for all \( 1 \leq j \leq m \), so in order to establish the required result, we need only to show that

\[
P \left( \sup_{0 \leq t \leq T} \|Z(t)\| > K \right) \rightarrow 0
\]  

(8)

as \( \nu^j \rightarrow 0 \) for all \( 1 \leq j \leq m \), where we define \( \nu_j = \nu_j(A) \) where \( A = (-\infty, -1] \cup [1, \infty) \). Using the Cauchy–Schwarz inequality for sums, we have

\[
P \left( \sup_{0 \leq t \leq T} \|Z(t)\| > K \right) = P \left( \sup_{0 \leq t \leq T} \|Z(t)\|^2 > K^2 \right)
\]  

\[
\leq \left( \sum_{1 \leq i \leq m} \sup_{0 \leq t \leq T} |Z_i(t)|^2 \right) \geq K^2
\]  

\[
\leq \left( \sup_{0 \leq t \leq T} \max_{1 \leq i \leq m} |Z_i(t)| > K^2 \right)
\]  

\[
= \left( \sup_{0 \leq t \leq T} \max_{1 \leq i \leq m} |Z_i(t)| > K \right)
\]

(9)

and so our goal is reached if we can prove that for all \( 1 \leq i \leq d, T \geq 0, K > 0 \)

\[
P \left( \sup_{0 \leq t \leq T} |Z_i(t)| > K \right) \rightarrow 0 \text{ as } \max_{1 \leq j \leq m} \nu_j \rightarrow 0.
\]

Define \( h_i = \max_{1 \leq j \leq m} \sqrt{H_j^i} \), then by (3)

\[
|Z_i(t)| \leq \max_{1 \leq j \leq m} \sup_{0 \leq t \leq T} |c_j^i(x_{t-})| \sum_{j=1}^m Q_j(t) \leq h_i \sum_{j=1}^m Q_j(t)
\]

where \( Q_j(t) = \int_0^t \int_{|y| \geq 1} |y|^j N^j(t, dy^j) \), for \( 1 \leq j \leq m, t \geq 0 \). We use the elementary inequality \( P(Y > K) \geq P(X > K) \) for random variables \( Y \geq X \geq 0 \) to see that

\[
P \left( \sup_{0 \leq t \leq T} |Z_i(t)| > K \right) \leq \left( \sup_{0 \leq t \leq T} \sum_{j=1}^m Q_j(t) > K \right)
\]  

\[
\leq \left( \sum_{j=1}^m \sup_{0 \leq t \leq T} Q_j(t) > K \right)
\]

(10)

where the second inequality follows from the fact that for random variables \( X_1, \ldots, X_m \), \( P(\max_{1 \leq j \leq m} |X_j| > K) \leq \sum_{j=1}^m P(|X_j| > K/m) \).
Hence, to establish (9), it is sufficient to prove that for each $1 \leq j \leq d$, $L > 0$

$$P \left( \sup_{0 \leq t \leq T} Q_j(t) > L \right) \rightarrow 0 \text{ as } \nu_j \rightarrow 0. \quad (10)$$

It is shown in [1, Ch. 2] that $Q_j = (Q_j(t), t \geq 0)$ is a compound Poisson process and that we can write $Q_j(t) = \sum_{n=1}^{N_j(t)} W_{j,n}$ where $(W_{j,n}, n \in \mathbb{N})$ is a sequence of nonnegative independent identically distributed (i.i.d.) random variables having common law

$$p_{W_{j,n}}(B) = \frac{\nu_j(-B \cap [(-\infty, -1)]) + \nu_j(B \cap [1, \infty])}{\nu_j}$$

and $(N_j(t), t \geq 0)$ is an independent Poisson process having intensity $\nu_j$. It follows that $\sup_{0 \leq t \leq T} Q_j(t) = Q_j(T)$ since $(N_j(t), t \geq 0)$ is nondecreasing.

The probability law of $Q_j(T)$ is

$$\mu_j(B) = \sum_{n=0}^{\infty} e^{-T \nu_j} \frac{T^n \nu_j^n}{n!} p_{W_{j,n}}(B)$$

$$= \sum_{n=0}^{\infty} e^{-T \nu_j} \frac{T^n \nu_j^n}{n!} \tilde{\nu}_j^n(B)$$

(see, e.g., [4, Ch. VI, Sec. 4]) where $n$ denotes the $n$th convolution power and $\tilde{\nu}_j(n) = \nu_j(-B \cap [(-\infty, -1)]) + \nu_j(B \cap [1, \infty])$.

It is easy to see that for all $n \in \mathbb{N}$, $\tilde{\nu}_j^n(B) \rightarrow 0$ as $\nu_j \rightarrow 0$ and so by dominated convergence it follows that $\mu_j(B) \rightarrow 0$ as $\nu_j \rightarrow 0$. We obtain (10) when we take $B = (L, \infty)$. \hfill \Box

SR follows from the result of Theorem 1 by the argument of Theorem 1 in [5]. The same arguments allow us to extend Lemma 2 of [5] to general Lévy noise and hence obtain SR for spiking neuron models. We remark that the condition that the $I^*_j$’s are independent Lévy processes, which is built into the model in [5], can be dropped and the results of this paper then extend easily to the case where $L_t$ is an arbitrary $\mathbb{R}^m$-valued Lévy process.

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