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Localizability and Distributed Protocols for Bearing-Based Network Localization in Arbitrary Dimensions

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Abstract
This paper addresses the problem of bearing-based network localization, which aims to localize all the nodes in a static network given the locations of a subset of nodes termed anchors and inter-node bearings measured in a common reference frame. The contributions of the paper are twofold. Firstly, we propose necessary and sufficient conditions for network localizability with both algebraic and rigidity theoretic interpretations. Secondly, we propose and analyze a linear distributed protocol for bearing-based network localization. One novelty of our work is that the localizability analysis and localization protocol are applicable to networks in arbitrary dimensional spaces.

Key words: Sensor network, Network localizability, Distributed localization, Bearing rigidity, Bearing Laplacian.

1 Introduction

Distributed localization of sensor networks is a core problem in many multi-agent coordination tasks. Network localizability and distributed protocols are two fundamental problems for any network localization problems. Network localizability characterizes whether or not a network can be possibly localized given the anchor locations and inter-neighbor relative measurements. According to the types of the relative measurements used for localization, the existing works can be divided into three classes: distance-based, bearing-based, and position-based. Distance-based network localization has been studied extensively so far (see [1–4] and the references therein). The analysis of the localizability in distance-based network localization relies heavily on the distance rigidity theory. More recently, bearing-based network localization has also attracted extensive research attention [5–11]. The analysis of the localizability in bearing-based network localization relies on the analogous bearing rigidity theory [12–15]. Finally, position-based network localization, where the inter-neighbor distance and local bearing measurements are used together for network localization, has been studied in [16] by using a complex graph Laplacian.

Although bearing-based network localization has been studied by many researchers, the two fundamental problems, network localizability and distributed protocols, have not yet been fully explored. Very recently, a necessary and sufficient condition for network localizability was proposed in [10, Thm 15] based on the notion of a stiffness matrix. This condition is applicable only to networks in two-dimensional spaces. The existing protocols for bearing-based network localization are also mainly applicable to networks in two-dimensional ambient spaces [6–11]. General results of localizability or distributed protocols for bearing-based network localization in three and higher dimensional spaces are still lacking. The main contributions of our work are summarized below.

(a) We formulate the problem of bearing-based network localization in arbitrary dimensions as a linear least-squares optimization problem. A special matrix termed the bearing Laplacian, which can be viewed as a matrix-weighted graph Laplacian, emerges as a key part in the least-squares formulation.

(b) By solving the least-squares problem, we propose necessary and sufficient conditions for network localizability with both algebraic and rigidity theoretic interpretations. These conditions not only provide numerical ways to examine the localizability of a given network but also provide intuitions on what a localizable network looks like.

(c) We then propose a distributed linear localization protocol. It is proved that the protocol can globally localize a network if and only if the network is localizable. The sensitivity of the protocol to constant measurement errors is also analyzed.

The rest of the paper is organized as follows. Section 2 presents the linear least-squares formulation of the bearing-based network localization problem. Section 3
explores the properties of the bearing Laplacian matrix. Section 4 presents necessary and sufficient conditions for network localizability. Section 5 proposes and analyzes a linear distributed localization protocol. Conclusions are drawn in Section 6.

Notations: Given $A_i \in \mathbb{R}^{p \times q}$ for $i = 1, \ldots, n$, denote $\text{diag}(A_i) \triangleq \text{blkdiag}(A_1, \ldots, A_n) \in \mathbb{R}^{np \times nq}$. Let $\| \cdot \|$ be the Euclidean norm of a vector or the spectral norm of a matrix, and $\otimes$ be the Kronecker product. Denote $I_d \in \mathbb{R}^{d \times d}$ as the identity matrix, and $1_d \triangleq [1, \ldots, 1]^T \in \mathbb{R}^d$. Let $\text{Null}(\cdot)$ and $\text{Range}(\cdot)$ be the null space and range space of a matrix, respectively.

2 Bearing-Based Network Localization

In this section, the problem of bearing-based network localization is formally stated and then formulated as a linear least-squares problem. Central to this problem is the notion of localizability, which is formally defined here.

2.1 Problem Statement

Consider a network of $n$ stationary nodes in $\mathbb{R}^d$ ($n \geq 2$ and $d \geq 2$). Assume no two nodes are collocated. Let $p_i \in \mathbb{R}^d$ be the location of node $i$ ($i = 1, \ldots, n$). Define the edge vector and the bearing between nodes $i$ and $j$ as

$$e_{ij} \triangleq p_j - p_i, \quad g_{ij} \triangleq \frac{e_{ij}}{\|e_{ij}\|}.$$

The unit vector $g_{ij}$ represents the relative bearing of $p_j$ with respect to $p_i$. Note $e_{ij} = -e_{ji}$ and $g_{ij} = -g_{ji}$. Suppose the locations of $n_a$ anchor nodes are already given and the locations of the remaining $n_f$ follower nodes are to be estimated ($n_a + n_f = n$). Denote $V_a = \{1, \ldots, n_a\}$, $V_f = \{n_a + 1, \ldots, n\}$, and $V = V_a \cup V_f$. Denote $p_0 = [p_1^T, \ldots, p_{n_a}^T]^T \in \mathbb{R}^{d n_a}$, $p_f = [p_{n_a+1}^T, \ldots, p_n^T]^T \in \mathbb{R}^{d n_f}$, and $p = [p_0^T, p_f^T]^T \in \mathbb{R}^{d n}$.

Suppose each node has the bearing-only sensing capabilities. The sensing topology of the network defines a graph $G = (V, E)$ where $E \subset V \times V$. Denote $(i, j)$ as the directed edge with node $i$ as the tail and node $j$ as the head. The directed edge $(i, j) \in E$ indicates that node $i$ can “see” node $j$; that is node $i$ can measure the relative bearings $g_{ij}$ of node $j$. Node $j$ is called the neighbor of node $i$ if $(i, j) \in E$, and $N_i' \triangleq \{j \in V |(i, j) \in E\}$ is the neighborhood of node $i$. We assume a global orientation that can be sensed by all the nodes, and thus all measured bearings can be expressed with respect to this common orientation. The global orientation means a common north for the two-dimensional space, and a common northeast-down reference for the three-dimensional space. Finally, let $\mathcal{G}(p)$ denote the network that is the graph $G$ with each vertex $i \in V$ mapped to the point $p_i$.

The problem of bearing-based network localization is formally stated below.

**Problem 1 (Bearing-Based Network Localizability)**

A network $\mathcal{G}(p)$ is called bearing-based localizable if the true network location $p$ is the unique solution to (1).

Locализируемость - это фундаментальная характеристика направленных сетей. Сеть должна быть локализуемой в порядке, чтобы иметь возможность быть локализована с помощью разнородных или централизованных протоколов. Нотация локализуемости иллюстрируется примером на рисунке 1. В этом примере, сеть на рисунке 1(a) является истинной сетью. Сеть на рисунке 1(b) имеет те же направления и узловые местоположения, что и истинная сеть. В результате, обе сети из рисунка 1(a)-(b) являются решениями (1) и дальше сети не локализуемы по определению 1.

For the sake of simplicity, we assume that the graph $\mathcal{G}$ is undirected, which means $(i, j) \in E \iff (j, i) \in E$. If the graph is directed, suppose $(i, j) \in E$ but $(j, i) \notin E$. We can always add the edge $(j, i)$ to $E$ to convert the directed graph to an undirected one. The directed edges $(i, j)$ and $(j, i)$ imply two equations $(\hat{p}_j - \hat{p}_i)/\|\hat{p}_j - \hat{p}_i\| = g_{ij}$ and $(\hat{p}_i - \hat{p}_j)/\|\hat{p}_i - \hat{p}_j\| = g_{ij}$, respectively. The two equations are equivalent because $g_{ij} = -g_{ji}$. As a result, adding the edge $(j, i)$ does not affect the solutions to (1).

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Fig. 1. An illustration of the notion of localizability. Suppose the network in (a) is the true network. The networks in (a) and (b) satisfy the nonlinear equations in (1). The networks in (a), (b), and (c) satisfy the linear equations in (2).
2.2 Reformulation as a Least-Squares Problem

In order solve the nonlinear equations in (1), we derive a companion system of linear equations. In this direction, we first introduce a useful orthogonal projection operator. For any nonzero vector \( x \in \mathbb{R}^d \) (\( d \geq 2 \)), define the orthogonal projection operator \( P : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d} \) as

\[
P(x) \triangleq I_d - \frac{x x^T}{\|x\|}.
\]

For notational simplicity, denote \( P_x \triangleq P(x) \). The matrix \( P_x \) geometrically projects any vector onto the orthogonal compliment of \( x \). It can be easily verified that \( P_x^2 = P_x \), \( P_x^2 = P_x \), \( \text{Null}(P_x) = \{x\} \), and the eigenvalues of \( P_x \) are \( \{0, 1^{(d-1)}\} \).

Consider now the projection matrix, \( P_{g_{ij}} = I_d - g_{ij}g_{ij}^T \), associated with the bearing \( g_{ij} \). By multiplying \( P_{g_{ij}} \) on both sides of the first equation in (1), the nonlinear algebraic problem (1) is converted to a system of linear equations,

\[
\begin{align*}
P_{g_{ij}} \left( \hat{p}_i - p_i \right) &= 0, \quad \forall (i, j) \in \mathcal{E}, \\
\hat{p}_i &= p_i, \quad \forall i \in \mathcal{V}_a.
\end{align*}
\]

The linear equations in (2) are not equivalent to the nonlinear equations in (1) in general. But we next show that the two sets of equations are equivalent when the true network localization is the unique solution.

Lemma 1 Let \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \) denote the set of all solutions satisfying (1) and (2), respectively. Then

(a) \( \{p\} \subseteq \mathcal{X}_1 \subseteq \mathcal{X}_2 \);
(b) \( \{p\} = \mathcal{X}_1 \) if and only if \( \{p\} = \mathcal{X}_2 \).

Proof. (a) Since the true network location \( p \) is always a solution to (1) and (2), we know \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \) are nonempty and \( \{p\} \subseteq \mathcal{X}_1 \) and \( \{p\} \subseteq \mathcal{X}_2 \). Since (2) is obtained by multiplying (1) by \( P_{g_{ij}} \), we know any solution to (1) is also a solution to (2), showing \( \mathcal{X}_1 \subseteq \mathcal{X}_2 \).

(b) (Sufficiency) Suppose \( \{p\} = \mathcal{X}_2 \). It then follows from \( \{p\} \subseteq \mathcal{X}_1 \subseteq \mathcal{X}_2 \) that \( \{p\} = \mathcal{X}_1 \). (Necessity) Suppose \( \{p\} = \mathcal{X}_1 \). We now prove \( \{p\} = \mathcal{X}_2 \) by contradiction. Assume \( p' \in \mathcal{X}_2 \) and \( p' \neq p \). Let \( \delta p \triangleq p' - p \) and define

\[
p'' \triangleq p + k\delta p, \quad k \in \mathbb{R}.
\]

We next show that \( p'' \in \mathcal{X}_1 \) when \( |k| \) is sufficiently small, leading to a contradiction. Since \( p, p' \in \mathcal{X}_2 \), we know \( p'' \in \mathcal{X}_2 \) for all \( k \in \mathbb{R} \) by (3). As a result, for any \( k \in \mathbb{R} \) and \( (i, j) \in \mathcal{E} \), we have \( P_{g_{ij}}(p'' - p') = 0 \) which implies either \( (p''_j - p'_j)/\|p''_j - p'_j\| = g_{ij} \) or \( (p''_j - p'_j)/\|p''_j - p'_j\| = -g_{ij} \). Since \( p''_j - p'_j = (p_j - p_i) + k(\delta p_j - \delta p_i) \) according to (3), it is obvious that when \( |k| \) is sufficiently small,

the entries of \( p''_j - p'_j \) have the same signs as those of \( p_j - p_i \), and consequently \( (p''_j - p'_j)/\|p''_j - p'_j\| = (p_j - p_i)/\|p_j - p_i\| = g_{ij} \). Note that when any entry of \( p_j - p_i \) is zero, the corresponding entry of \( \delta p_j - \delta p_i \) is also zero because \( \delta p_j - \delta p_i \) is parallel to \( p_j - p_i \). To conclude, \( p'' \) is another solution other than \( p \) satisfying (1), which is a contradiction. \( \square \)

Lemma 1(b) indicates that the true network location \( p \) is the unique solution to (1) if and only if \( p \) is the unique solution to (2). Thus we can study the localizability by analyzing the linear system (2). The linear system of equations in (2) can be rewritten as the following linear least-squares problem,

\[
\text{minimize}_{\hat{p} \in \mathbb{R}^{dn}} \quad J(\hat{p}) = \frac{1}{2} \sum_{i \in \mathcal{V}_a} \sum_{j \in \mathcal{N}_i} \|P_{g_{ij}}(\hat{p}_i - \hat{p}_j)\|^2, \quad (4)
\]

subject to \( \hat{p}_i = p_i, \quad i \in \mathcal{V}_a \).

Since any minimizer with the objective function as zero is the solution to (2), we now successfully formulate the localizability problem as the above least-squares problem. The rest of the paper is dedicated to studying two properties of the least-squares problem. The first is to determine when the true location \( p \) is the unique global minimizer of (4) (i.e., when the network is localizable), and the second is how to obtain \( p \) in a distributed manner (i.e., what the distributed localization protocol is).

3 The Bearing Laplacian Matrix

In this section, we show that a new important matrix, termed bearing Laplacian, emerges in the least-squares formulation. The useful properties of the bearing Laplacian that will be used throughout the paper are explored.

Since the underlying graph \( \mathcal{G} \) is undirected, the objective function in (4) can be expressed in a quadratic form,

\[
J(\hat{p}) = \hat{p}^T \mathcal{B}(\mathcal{G}(p)) \hat{p},
\]

where \( \mathcal{B}(\mathcal{G}(p)) \in \mathbb{R}^{dn \times dn} \) and its \( ij \)th subblock matrix is

\[
\begin{cases}
0_{d \times d}, & i \neq j, (i, j) \notin \mathcal{E}, \\
-P_{g_{ij}}, & i \neq j, (i, j) \in \mathcal{E}, \\
\sum_{k \in \mathcal{N}_i} P_{g_{ik}}, & i = j, i \in \mathcal{V}.
\end{cases}
\]

For notational simplicity, we write \( \mathcal{B}(\mathcal{G}(p)) \) as \( \mathcal{B} \) in the sequel. The matrix \( \mathcal{B} \) has a structure reminiscent of the weighted graph Laplacian matrix. Since \( \mathcal{B} \) indicates not only the topology of the network but also the neighbor bearings, it is referred to as bearing Laplacian in this paper.

The bearing Laplacian has an intimate connection to the bearing rigidity properties of the network. Preliminaries to the bearing rigidity theory, originally proposed in [15],
are given in Appendix A. Here we would like to highlight two important notions from this theory. The first is the notion of infinitesimal bearing motions. Loosely speaking, infinitesimal bearing motions are motions of the nodes that preserve inter-neighbor bearings. For example, for the network in Figure 1(a), the bearings can be preserved when the nodes 3 and 4 move in the horizontal direction to the right. A network always has two kinds of trivial infinitesimal bearing motions—they are the translational and scaling motions of the entire network. A network is infinitesimally bearing rigid if all its infinitesimal bearing motions are trivial. One important property of an infinitesimally bearing rigid network is that its shape can be uniquely determined by the inter-neighbor bearings.

We next give the basic properties of the bearing Laplacian matrix. We also show that the bearing Laplacian matrix is a powerful tool for characterizing the bearing rigidity of a network.

**Lemma 2** For a network \( G(p) \) with undirected graph \( G \), the bearing Laplacian \( B \) satisfies the following:

(a) \( B \) is symmetric positive semi-definite;
(b) \( \text{Rank}(B) \leq dn - d - 1 \) and \( \text{Null}(B) \supseteq \text{span} \{ 1 \otimes I_d, p \} \);
(c) \( \text{Rank}(B) = dn - d - 1 \) and \( \text{Null}(B) = \text{span} \{ 1 \otimes I_d, p \} \)

if and only if \( G(p) \) is infinitesimally bearing rigid.

**Proof.** Assign an arbitrary orientation to each undirected edge and label the edge vectors and bearings for the directed edges as \( \{ e_k \}_{k=1}^m \) and \( \{ g_k \}_{k=1}^m \), respectively. Then the bearing Laplacian \( B \) can be expressed as

\[
B = \tilde{H}^T \text{diag}(P_{g_k}) \tilde{H} = \mathcal{R}^T \mathcal{R},
\]

where \( \mathcal{R} = \text{diag}(\| e_k \| I_d) R_B \) where \( R_B \) is the bearing rigidity matrix (see Lemma 7 in Appendix A). As a result, the matrix \( \mathcal{R} \), and hence \( B \), have exactly the same rank and null space as \( R_B \). Then the results in (b) and (c) follows immediately from Lemma 7 and Theorem 7 as given in Appendix A. \( \square \)

Since the nodes in the network are partitioned into anchors and followers, it will be useful to partition the corresponding bearing Laplacian as

\[
B = \begin{bmatrix}
B_{aa} & B_{af} \\
B_{fa} & B_{ff}
\end{bmatrix},
\]

where \( B_{aa} \in \mathbb{R}^{dn_a \times dn_a}, B_{af} = B_{fa}^T \in \mathbb{R}^{dn_a \times dn_f}, \) and \( B_{ff} \in \mathbb{R}^{dn_f \times dn_f} \).

**Lemma 3** For any network \( G(p) \) with undirected graph \( G \), the subblock matrix \( B_{ff} \) is symmetric positive semi-definite and satisfies \( B_{ff} p_f + B_{fa} p_a = 0 \).

**Proof.** For any nonzero \( x \in \mathbb{R}^{dn_f} \), denote \( \tilde{x} = [0, x^T]^T \in \mathbb{R}^{dn} \). Since \( B \geq 0 \), we have \( x^T B_{ff} x = x^T B \tilde{x} \geq 0 \). As a result \( B_{ff} \) is positive semi-definite. Since \( p \in \text{Null}(B) \) as suggested by Lemma 2, we have \( B p = 0 \) which further implies \( B_{fa} p_a + B_{ff} p_f = 0 \). \( \square \)

4 Analysis of Network Localizability

In this section, we analyze the localizability of networks in arbitrary dimensions. We first prove two necessary and sufficient conditions for network localizability from algebraic and rigidity perspectives, respectively. We then present more necessary and/or sufficient conditions which can give more intuition on what localizable networks look like. First of all, we derive the optimality condition for the least-squares problem (4).

**Lemma 4** For the least-squares problem (4), any minimizer \( \hat{p}_f \) is also a global minimizer and satisfies

\[
B_{ff} \hat{p}_f + B_{fa} p_a = 0.
\]

**Proof.** By substituting \( \hat{p}_a = p_a \) into the objective function \( \hat{J}(\hat{p}) = \hat{p}^T B \hat{p}_a \), the constrained optimization problem (4) can be converted to the unconstrained problem

\[
\min_{\hat{p}_f \in \mathbb{R}^{dn_f}} \hat{J}(\hat{p}_f) = \hat{p}_f^T B_{ff} \hat{p}_f + 2 p_a^T B_{af} \hat{p}_f + p_a^T B_{aa} p_a.
\]

Any minimizer must satisfy \( \nabla_{\hat{p}_f} \hat{J}(\hat{p}_f) = B_{ff} \hat{p}_f + B_{fa} p_a = 0 \). Now suppose \( \hat{p}_f \) is a minimizer and satisfies \( B_{ff} \hat{p}_f + B_{fa} p_a = 0 \). By comparing with \( B_{ff} \hat{p}_f + B_{fa} p_a = 0 \) as shown in Lemma 3, we know \( \hat{p}_f = p_f + x \) where \( x \in \text{Null}(B_{ff}) \). Let \( \hat{p}^* = [p^*_f, (\hat{p}_f)^T]^T \) and \( \hat{x} = [0, x^T]^T \in \mathbb{R}^{dn} \). Since \( \hat{p}_f = p_f + x \) and \( B \tilde{x} = 0 \), we have \( \hat{J}(\hat{p}^*) = (p_f^*)^T B \tilde{x}^* + (p + \hat{x})^T B(p + \tilde{x}) = x^T B \tilde{x} = x^T B_{ff} x = 0 \). As a result, the objective function equals zero at every minimizer. \( \square \)

The linear equations in (2) hold if and only if the objective function in the least-squares problem (4) is minimized to zero; this is a direct consequence of the first-order optimality conditions associated with (4). Thus the equivalence between (2) and (4) is formally established. We are now ready to present the necessary and sufficient condition for localizability.

**Theorem 1** (Algebraic Condition for Localizability)
A network \( G(p) \) is localizable if and only if the matrix \( B_{ff} \) is nonsingular. When the network is localizable, the true locations of the followers can be calculated by \( p_f = -B_{ff}^{-1} B_{fa} p_a \).

**Proof.** By Lemma 4, a network is localizable if and only if the true network location \( p \) is the unique minimizer of the least-squares problem (4). Since any minimizer
must satisfy $B_{ff}\hat{\delta}_f + B_{fa}\delta_a = 0$, it is obvious that the minimizer is unique if and only if $B_{ff}$ is nonsingular. When $B_{ff}$ is nonsingular, we have $\hat{\delta}_f^* = -B_{ff}^{-1}B_{fa}\delta_a$, whose value equals the true location $p_f$ according to Lemma 3.

Theorem 1 establishes the equivalence between the localizability and the nonsingularity of $B_{ff}$. A question that immediately follows Theorem 1 is what kind of networks have nonsingular $B_{ff}$. We next propose a necessary and sufficient condition from the bearing rigidity point of view. This rigidity condition is mathematically equivalent to the algebraic condition, but it gives more intuition on what localizable networks look like.

**Theorem 2 (Rigidity Condition for Localizability)**

A network $G(p)$ is localizable if and only if every infinitesimal bearing motion involves at least one anchor; that is, for any nonzero infinitesimal bearing motion $\delta p = [\delta p_a^T, \delta p_f^T]^T \in \text{Null}(B)$, the motion $\delta p_a$ corresponding to the anchors must be nonzero.

**Proof.** We only need to show that $B_{ff}$ is singular if and only if there exists nonzero $\delta p \in \text{Null}(B)$ with $\delta p_a = 0$. (Necessity) Suppose $B_{ff}$ is singular. Then there exists $x \in \mathbb{R}^{dn}$ such that $B_{ff}x = 0$. Let $\delta p = [0, x^T]^T \in \mathbb{R}^{dn}$. Then $\delta p^TB\delta p = x^TB_{ff}x = 0$. Hence $\delta p \in \text{Null}(B)$ and $\delta p_a = 0$. (Sufficiency) Suppose there exists $\delta p \in \text{Null}(B)$ satisfying $\delta p_a = 0$ and $\delta p_f \neq 0$. Then $\delta p^TB_{ff}\delta p_f = \delta p^TB\delta p = 0$, which implies that $B_{ff}$ is singular.

The intuition behind Theorem 2 is as follows. Any infinitesimal bearing motion (i.e., bearing-preserved motion) would imply multiple false networks that have exactly the same bearings as the true network. Only if the infinitesimal bearing motion involves at least one anchor, the false networks can be ruled out as solutions to (1) since they do not satisfy the anchor constraints; otherwise, the false networks cannot be distinguished from the true network.

Examples are given in Figure 2 and Figure 3 to illustrate Theorem 2. Figure 2 shows examples of non-localizable networks. These networks are not localizable because each of them has infinitesimal bearing motions that only involve the followers (see those marked by red arrows). Figure 3 shows examples of localizable networks. The networks in Figure 3(a)-(f) are obtained by modifying the networks in Figure 2, which suggests that a non-localizable network can be made localizable by adding extra edges or selecting different anchors. It is worth noting that the networks in Figure 3(c)-(g) are not infinitesimally bearing rigid yet they are localizable. As a result, infinitesimal bearing rigidity is not necessary to guarantee localizability.

Up to this point, we have presented two necessary and sufficient localizability conditions. We next utilize the two conditions to examine some specific problems more closely. The first is to examine how many anchors are required to ensure the localizability of a network.

**Corollary 1** If a network $G(p)$ is localizable, then

$$n_a \geq \frac{\dim(\text{Null}(B))}{d} > 1.$$  

**Proof.** Let $k = \dim(\text{Null}(B))$ and $N \in \mathbb{R}^{dn \times k}$ be a basis matrix of $\text{Null}(B)$ which means Range($N$) = $\text{Null}(B)$. Then any nonzero $\delta p \in \text{Null}(B)$ can be expressed as $\delta p = Nx$, where $x \in \mathbb{R}^k$, $x \neq 0$. Partition $N$ and express $Nx$ as $\delta p = Nx = \begin{bmatrix} N_a x \\ N_f x \end{bmatrix}$, where $N_a \in \mathbb{R}^{dn_a \times k}$. According to Theorem 2, the network is localizable if and
only if $N_a x \neq 0, \forall x \in \mathbb{R}^k, x \neq 0$. As a result, the matrix $N_a$ must have full column rank, which requires $N_a$ to be a tall matrix with $\text{dim}(N_a) \geq k = \text{dim}(\text{Null}(B))$. Since $\text{dim}(\text{Null}(B)) \geq d + 1$ according to Lemma 2, we have $n_a \geq \text{dim}(\text{Null}(B)) / d \geq (d + 1) / d > 1$. □

A simple but important fact suggested by Corollary 1 is that any localizable network must have at least two anchors. Similar conclusions have already been obtained in the existing studies for networks in the two-dimensional space [7–10]. Another important fact, which is suggested by Corollary 1 but has not been reported in the literature, is that more anchors are required to ensure the localizability when $\text{dim}(\text{Null}(B))$ increases. The quantity $\text{dim}(\text{Null}(B))$ can be viewed as a measure of the “degree of bearing rigidity” as $\text{dim}(\text{Null}(B))$ reaches the smallest value $d + 1$ when the network is infinitesimally bearing rigid as shown in Lemma 2. As a result, the intuition behind the second fact is that more anchors are required to ensure the localizability when the network is “less” bearing rigid (i.e., $\text{dim}(\text{Null}(B))$ is large).

We next present another three localizability conditions, two of which are sufficient and the other is both necessary and sufficient. These conditions are important because they indicate the explicit connection between the localizability and infinitesimal bearing rigidity. Before presenting the conditions, we need to first define the notion of augmented network.

**Definition 2 (Augmented Network)** Given a network $G(p)$ with $G = (V, E)$, denote by $\bar{G}(p)$ an augmented network with $\bar{G} = (V, \bar{E})$ where $\bar{E} = \bar{E} \cup \{(i, j) : i, j \in V_a\}$.

The augmented network $\bar{G}(p)$ is obtained from $G(p)$ by connecting every pair of anchors. If the anchors are already connected in $G(p)$, then $\bar{G}(p)$ is the same as $G(p)$. It should be noted that adding or deleting the edge between any pair of anchors only changes $B_{aa}$ but not $B_{ff}$. As a result, $G(p)$ and $\bar{G}(p)$ have exactly the same $B_{ff}$ and hence they are localizable or nonlocalizable simultaneously. The next two sufficient conditions connect the notions of localizability and infinitesimal bearing rigidity.

**Corollary 2** When $n_a \geq 2$, if $\bar{G}(p)$ is infinitesimally bearing rigid, then $G(p)$ is localizable.

**Proof.** We will first use Theorem 2 to prove the localizability of $\bar{G}(p)$. Then the localizability of $\bar{G}(p)$ immediately follows because $\bar{G}(p)$ and $G(p)$ have the same localizability. Let $B$ be the bearing Laplacian for $G(p)$. Since $\bar{G}(p)$ is infinitesimally bearing rigid, we have $\text{Null}(B) = \text{span}\{1 \otimes I_d, p\}$ by Lemma 2. As a result, any infinitesimal bearing motion $\delta p \in \text{Null}(B)$ can be expressed as a linear combination of $1 \otimes I_d$ and $p$. Since no two anchors collocate, there does not exist a linear combination of $1 \otimes I_d$ and $p$ leading to $\delta p = 0$ if $n_a \geq 2$. Then $\bar{G}(p)$ is localizable according to Theorem 2. □

**Corollary 3** When $n_a \geq 2$, if $G(p)$ is infinitesimally bearing rigid, then $\bar{G}(p)$ is localizable.

**Proof.** Similar to Corollary 2. □

The intuition behind Corollary 3 is as follows. If a network is infinitesimally bearing rigid, then it can be uniquely determined up to a translation and a scaling factor by the bearings. Since the translational and scaling ambiguity can be further eliminated by the anchor constraints, the entire network can be fully determined and hence localizable. It is notable that Corollary 3 is more restrictive than Corollary 2 because it requires $\bar{G}(p)$ to be infinitesimally bearing rigid whereas Corollary 2 merely requires $G(p)$ to be. To illustrate, the networks as shown in Figure 3(c)–(f) are localizable. For each of them, the augmented network $\bar{G}(p)$ is infinitesimally bearing rigid but $G(p)$ is not. Finally, Corollary 2 can be viewed as a generalization of the result [10, Cor 10] which is applicable only to two-dimensional cases.

As suggested by Corollary 2, the condition of the infinitesimal bearing rigidity of $G(p)$ is sufficient to ensure the localizability of $\bar{G}(p)$. An important yet unexplored problem is whether or not the condition is also necessary. In the case of $n_a \geq 3$, the condition is sufficient but not necessary. A counterexample is given in Figure 3(g), where $G(p)$ is localizable but $\bar{G}(p)$ is not infinitesimally bearing rigid since the three anchors are collinear. However, in the case of $n_a = 2$, we can prove that the condition is both necessary and sufficient.

**Theorem 3** When $n_a = 2$, a network $G(p)$ is localizable if and only if the augmented network $\bar{G}(p)$ is infinitesimally bearing rigid.

**Proof.** The sufficiency has already been proved in Corollary 2. We next prove the necessity by contradiction. Assume $G(p)$ is localizable but $\bar{G}(p)$ is not infinitesimally bearing rigid. Then $\bar{G}(p)$ has a nontrivial infinitesimal bearing motion $\delta p$ which is not in span $\{1 \otimes I_d, p\}$. Write $\delta p = [\delta p_1, \delta p_2, (\ast)]^T$, where $\delta p_1, \delta p_2 \in \mathbb{R}^d$ corresponds to the two anchors. Because the infinitesimal motion $\delta p$ preserves all the bearings including the bearing between $p_1$ and $p_2$, we know that the vector $\delta p_1 - \delta p_2$ is parallel to $p_1 - p_2$. As a result, there exists a nonzero scalar $k$ such that $\delta p_1 - \delta p_2 = k(p_1 - p_2)$. Construct

$$\delta p' = \delta p + 1_n \otimes (kp_2 - \delta p_2) - kp \begin{bmatrix} \delta p_1 \\ \delta p_2 \\ (\ast) \end{bmatrix} + \begin{bmatrix} kp_2 - \delta p_2 \\ k(p_1 - p_2) \\ (\ast) \end{bmatrix} - \begin{bmatrix} k(p_1 - p_2) \\ k(p_2 - \delta p_2) \\ (\ast) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. $$

Since the first two entries of $\delta p'$ are zero, we know $\delta p'$ is an infinitesimal motion that only involves the followers. Thus, the network is not localizable by Theorem 2, which is a contradiction. □
5 Distributed Network Localization Protocols

In this section, we propose and analyze a linear distributed protocol for bearing-based network localization in arbitrary dimensions.

The global minimizer of the unconstrained optimization problem (5) can be obtained by the gradient decent protocol

\[ \dot{\hat{p}}_f(t) = -\nabla \hat{p}_f J(\hat{p}_f) = -B_{ff}\hat{p}_f(t) - B_{fa}p_a, \]  

whose elementwise expression is

\[ \dot{\hat{p}}_i(t) = -\sum_{j \in N_i} P_{gi}(\hat{p}_i(t) - \hat{p}_j(t)), \quad i \in V_f. \]  

where \( P_{gi} = I_d - g_{ij}g_{ij}^T \). Note the neighbor of the follower \( i \) can be either a follower or an anchor. Several remarks for protocol (7) are given below. First, the protocol is distributed because the localization of \( p_i \) only requires \( \{g_{ji}\}_{j \in N_i} \) and \( \{\hat{p}_j\}_{j \in N_i} \). In practical implementation, the bearings \( \{g_{ji}\}_{j \in N_i} \) can be measured by a bearing-only sensor such as a camera and the estimates \( \{\hat{p}_j\}_{j \in N_i} \) can be transmitted from the neighbors via wireless communication. All the bearings must be measured in a global reference frame. Second, the protocol has a clear geometric interpretation as shown in Figure 4.

![Figure 4. The geometric interpretation of protocol (7).](image)

The convergence of the protocol is characterized as below.

**Theorem 4** The distributed protocol (7) can globally localize the network \( \mathcal{G}(p) \) if and only if the network is localizable.

**Proof.** When \( B_{ff} \) is nonsingular (i.e., the network is localizable), the matrix \( -B_{ff} \) is Hurwitz. As a result, the linear time-invariant system (6) is stable and the state converges to the steady state value \( -B_{ff}^{-1}B_{fa}p_a \) which equals to the real follower location \( p_f \) according to Lemma 3. When \( B_{ff} \) is singular (i.e., the network is not localizable), the final estimate would depend on the initial estimate of the network location. \( \square \)

5.1 Sensitivity Analysis

Since the bearing measurements may be corrupted by errors in practice, it is meaningful to study the impact of constant measurement errors on the localization protocol (7). Denote the unit vector \( \tilde{g}_{ij} \in \mathbb{R}^d \) as the measurement of \( g_{ij} \). In the presence of bearing measurement errors, the localization protocol (6) becomes

\[ \dot{\hat{p}}_f(t) = -\tilde{B}_{ff}\hat{p}_f(t) - \tilde{B}_{fa}p_a, \]  

where \( \tilde{B}_{ff} \) and \( \tilde{B}_{fa} \) are obtained from \( B_{ff} \) and \( B_{fa} \) by replacing \( g_{ij} \) with \( \tilde{g}_{ij} \), respectively. The matrix \( \tilde{B}_{ff} \) may not be symmetric since \( \tilde{g}_{ij} \neq -\tilde{g}_{ji} \) in general.

We next analyze two problems regarding (8). The first is when \( \tilde{B}_{ff} \) is positive stable (i.e., all its eigenvalues have positive real parts) such that (8) is globally stable. If \( \tilde{B}_{ff} \) is positive stable, the final estimate given by (8) is

\[ \hat{p}_f^* = -\tilde{B}_{ff}^{-1}\tilde{B}_{fa}p_a. \]  

The second problem is how large the localization error \( \|\hat{p}_f^* - p_f\| \) is. To solve the two problems, define

\[ \Delta B_{ff} \triangleq \tilde{B}_{ff} - B_{ff}, \quad \Delta B_{fa} \triangleq \tilde{B}_{fa} - B_{fa}, \]  

as the perturbations of \( B_{ff} \) and \( B_{fa} \) caused by the bearing measurement errors. Let \( \theta_{ij} \in [0, \pi] \) be the angle between \( \tilde{g}_{ij} \) and \( g_{ij} \); that is \( \tilde{g}_{ij}^Tg_{ij} = \cos \theta_{ij} \). The angle \( \theta_{ij} \) represents the inconsistency between \( \tilde{g}_{ij} \) and \( g_{ij} \). This representation is valid for arbitrary dimensions. Note \( \theta_{ij} \neq \theta_{ji} \) in general. Define the total bearing measurement error for the followers as

\[ \epsilon \triangleq 2 \sum_{i \in V_f} \sum_{j \in N_i} \sin \theta_{ij}. \]

We next give lemmas to characterize the relationship between \( \epsilon \) and \( \Delta B_{ff}, \Delta B_{fa} \).

**Lemma 5** Denote by \( \theta \in [0, \pi] \) the angle between any two nonzero vectors \( x, y \in \mathbb{R}^d \) (i.e., \( x^Ty = \|x\|\|y\|\cos \theta \)). Then \( \|P_x - P_y\| = \sin \theta \).

**Proof.** See Appendix B. \( \square \)
Lemma 6 For a network \( \mathcal{G}(p) \) with arbitrary bearing measurements \( \{g_{ij}\}_{(i,j) \in E} \), it always holds that \( \|\Delta B_{ff}\| \leq \epsilon \) and \( \|\Delta B_{fa}\| \leq \epsilon/2 \).

Proof. Denote \( \Delta P_{g_{ij}} \triangleq P_{g_{ij}} - P_{g_{ij}} \), \( \forall (i,j) \in \mathcal{E} \). It then follows from Lemma 5 that \( \|\Delta P_{g_{ij}}\| = \sin \theta_{ij} \). Note \( \|\Delta B_{ff}\| = \sum_{j \in \mathcal{N}_{i} \cap \mathcal{V}_{j}} \|\Delta P_{g_{ij}}\| + \sum_{j \in \mathcal{V}_{j}} \sum_{j \in \mathcal{N}_{i} \cap \mathcal{V}_{j}} \|\Delta P_{g_{ij}}\| \leq 2 \sum_{j \in \mathcal{V}_{j}} \sum_{j \in \mathcal{N}_{i} \cap \mathcal{V}_{j}} \|\Delta P_{g_{ij}}\| = 2 \sum_{j \in \mathcal{V}_{j}} \sum_{j \in \mathcal{N}_{i} \cap \mathcal{V}_{j}} \sin \theta_{ij} = \epsilon \).

We now give an upper bound for the total bearing error \( \epsilon \) to ensure the positive stability of \( \hat{B}_{ff} \).

Theorem 5 Given a localizable network with \( B_{ff} \) nonsingular, the matrix \( \hat{B}_{ff} \) is positive stable if the total bearing error \( \epsilon \) satisfies

\[ \epsilon < \lambda_{\min}(B_{ff}), \tag{10} \]

where \( \lambda_{\min}(B_{ff}) \) is the minimum eigenvalue of \( B_{ff} \).

Proof. Since \( \|\Delta B_{ff}\| < \epsilon \) by Lemma 6, if (10) holds, we have \( \|\Delta B_{ff}\| < \lambda_{\min}(B_{ff}) = 1/\|B_{ff}^{-1}\| \), which further implies \( \|B_{ff}^{-1}\Delta B_{ff}\| < 1 \). Thus the spectral radius \( \rho(B_{ff}^{-1}\Delta B_{ff}) < 1 \) and hence the matrix \( (I + B_{ff}^{-1}\Delta B_{ff}) \) is nonsingular. As a result, \( \hat{B}_{ff} = B_{ff} + \Delta B_{ff} = B_{ff}(I + B_{ff}^{-1}\Delta B_{ff}) \) is nonsingular. Since \( \hat{B}_{ff} \) is obtained by perturbing \( B_{ff} \) and \( B_{ff} \) is positive stable, the nonsingularity of \( \hat{B}_{ff} \) implies the positive stability.

Theorem 5 suggests that a large \( \lambda_{\min}(B_{ff}) \) would give the network a large tolerance to bearing measurement errors.

We now study the localization error \( \|p_{f} - p_{a}\| \). An intuitive conclusion that can be immediately drawn from (9) and matrix perturbation theory is that the localization error would be sufficiently small when the bearing measurement errors are sufficiently small. We next give a specific upper bound on the localization error.

Theorem 6 The estimate \( \hat{p}_{f} = -B_{ff}^{-1}\hat{B}_{fa}p_{a} \) given in (9) satisfies \( \|\hat{p}_{f} - p_{f}\| \leq \frac{\epsilon}{\lambda_{\min}(B_{ff})-\epsilon} \left( \frac{1}{2}\|p_{a}\| + \|p_{f}\| \right) \).

Proof. See Appendix C.

In the last, we briefly discuss the impact of measurement errors in the anchors’ locations. Suppose the bearing measurements are accurate in this case. Then the final estimate given by protocol (7) becomes

\[ \hat{p}_{f} = -B_{ff}^{-1}B_{fa}(p_{a} + \Delta p_{a}), \]

where \( \Delta p_{a} \in \mathbb{R}^{d_{as}} \) denotes the anchor location error. Then the localization error is given by \( \|\Delta \hat{p}_{f} \triangleq \hat{p}_{f} - p_{f} = -B_{ff}^{-1}B_{fa}\Delta p_{a} \), which indicates that the anchor location errors propagate to the final localization error via a linear transformation. It is straightforward to show that a translational or scaling error in the anchor measurements would cause the same translational or scaling error in the localization of followers.

5.2 Simulation Examples

Two simulation examples are shown in Figure 5 to demonstrate the localization protocol (7). The network to be localized is a three-dimensional cubic network, which contains eight nodes and two of them are anchors and the other six are followers. The initial estimate, which is randomly generated, is given in Figure 5(a).

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A Preliminaries to Bearing Rigidity Theory

For a network $G(p)$, consider an oriented graph and express the edge vector and the bearing for the $k$th directed edge in the oriented graph, respectively, as $e_k$ and $g_k = e_k/\| e_k \|$ for $k \in \{1, \ldots, m \}$. Define the bearing function $F_B : \mathbb{R}^m \rightarrow \mathbb{R}^m$ as $F_B(p) \triangleq \begin{bmatrix} g_1^T, \ldots, g_m^T \end{bmatrix}^T$. The bearing rigidity matrix is defined as the Jacobian of the bearing function, $R_B(p) \triangleq \partial F_B(p)/\partial p \in \mathbb{R}^{m \times dn}$. Two important properties of the bearing rigidity matrix are given as below.

Lemma 7 ([15]) For any network $G(p)$, the bearing rigidity matrix satisfies $R_B = \text{diag}(P_{e_k}/\| e_k \|) I$. Rank($R_B$) $\leq$ $dn - d - 1$ and span $\{ I \otimes I_d, p \} \subseteq \text{Null}(R_B)$.

Let $\delta p$ be a variation of $p$. If $R_B(p) \delta p = 0$, then $\delta p$ is called an infinitesimal bearing motion of $G(p)$. A network always has two kinds of trivial infinitesimal bearing motions: translation and scaling of the entire network.

Definition 3 (Infinitesimal Bearing Rigidity) A network is infinitesimally bearing rigid if all the infinitesimal bearing motions are trivial.

The necessary and sufficient conditions for infinitesimal bearing rigidity are summarized as below.

Theorem 7 ([15]) For any network $G(p)$, the following statements are equivalent:

(a) $G(p)$ is infinitesimally bearing rigid;

(b) $G(p)$ can be uniquely determined up to a translation and a scaling factor by the inter-neighbor bearings;

(c) $\text{Rank}(R_B) = dn - d - 1$;

(d) $\text{Null}(R_B) = \{ I \otimes I_d, p \}$.

B Proof of Lemma 5

Proof. Here we only prove the case of $d = 3$. Without loss of generality, assume $x$ and $y$ are two unit vectors satisfying $\| x \| = \| y \| = 1$. Then, we have $P_x = I_d - xx^T, \ P_y = I_d - yy^T$, and hence $\| P_x - P_y \| = \| xx^T - yy^T \|$. There always exists an orthogonal matrix $U \in \mathbb{R}^{3 \times 3}$ such that the two vectors $x$ and $y$ can be orthogonally transformed to $UX = \begin{bmatrix} 1, 0, 0 \end{bmatrix}^T$ and $UY = [\cos \theta, \sin \theta, 0]^T$. Since the spectral norm is invariant to orthogonal matrices, we have

$$\| P_x - P_y \| = \| U(xx^T - yy^T)U^T \|$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \sin \theta \cos \theta & \sin^2 \theta \end{bmatrix} = \sin \theta \| Q \|,$$

where $Q = \begin{bmatrix} \sin \theta & -\cos \theta \\ -\cos \theta & -\sin \theta \end{bmatrix}$. It is easy to see $Q^TQ = I_2$ and hence $Q$ is an orthogonal matrix. Then, $\| P_x - P_y \| = \sin \theta \| Q \| = \sin \theta \| I \| = \sin \theta$. □

C Proof of Theorem 6

Proof. Recall $p_f = -B_{ff}^{-1}B_{fa}p_a$ and note $\hat{p}_f = -(B_{ff} + \Delta B_{ff})^{-1}(B_{fa} + \Delta B_{fa})p_a$. By [22, Eq. (25)], we have

$$(B_{ff} + \Delta B_{ff})^{-1} = B_{ff}^{-1} - B_{ff}^{-1}\Delta B_{ff}(I + B_{ff}^{-1}\Delta B_{ff})^{-1}B_{ff}^{-1},$$

substituting into $\hat{p}_f$ gives $\hat{p}_f = -B_{ff}^{-1}B_{fa}p_a - B_{ff}^{-1}\Delta B_{fa}p_a + B_{ff}^{-1}\Delta B_{ff}(I + B_{ff}^{-1}\Delta B_{ff})^{-1}B_{ff}^{-1}B_{fa}p_a + B_{ff}^{-1}\Delta B_{ff}(I + B_{ff}^{-1}\Delta B_{ff})^{-1}p_f$. It follows that

$$\| \hat{p}_f - p_f \| \leq \|(I + B_{ff}^{-1}\Delta B_{ff})^{-1}B_{ff}^{-1}B_{fa}p_a\|
+ \|B_{ff}^{-1}\Delta B_{ff}(I + B_{ff}^{-1}\Delta B_{ff})^{-1}p_f\|
\leq \|(I + B_{ff}^{-1}\Delta B_{ff})^{-1}\|\|B_{ff}^{-1}\|\|\Delta B_{fa}p_a\|\|
+ \|B_{ff}^{-1}\|\|\Delta B_{ff}\|\|I + B_{ff}^{-1}\Delta B_{ff}\|^{-1}\|p_f\|
= \|(I + B_{ff}^{-1}\Delta B_{ff})^{-1}\|\|B_{ff}^{-1}\|\|\Delta B_{fa}p_a\|\|
+ \|B_{ff}^{-1}\|\|\Delta B_{ff}\|\|p_f\|.$$

Substituting $\|\Delta B_{ff}\| \leq \epsilon$ and $\|\Delta B_{fa}\| \leq \epsilon/2$ as shown in Lemma 6, and $\| (I + B_{ff}^{-1}\Delta B_{ff})^{-1} \| \leq 1/(1 - \|B_{ff}^{-1}\|\|\Delta B_{ff}\|)$ by [23, Lemma 2.3.3] into the above inequality gives $\| \hat{p}_f - p_f \| \leq \|B_{ff}^{-1}\| \|B_{fa}p_a\| + \|\Delta B_{fa}p_a\|/\lambda_{\min}(B_{ff}).$ Substituting $\|B_{ff}^{-1}\| = 1/\lambda_{\min}(B_{ff})$ completes the proof. □
References


