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Almost Global Attractivity of a Synchronous Generator Connected to an Infinite Bus

Nikita Barabanov, Johannes Schiffer, Romeo Ortega and Denis Efimov

Abstract—The problem of deriving verifiable conditions for stability of the equilibria of a realistic model of a synchronous generator with constant field current connected to an infinite bus is studied in the paper. Necessary and sufficient conditions for existence and uniqueness of equilibrium points are provided. Furthermore, sufficient conditions for almost global attractivity are given. To carry out this analysis a new Lyapunov–like function is proposed to establish convergence of bounded trajectories, while the latter is proven using the powerful theoretical framework of cell structures pioneered by Leonov and Noldus.

I. INTRODUCTION

Today’s electrical power systems are very large, complex and highly nonlinear [1], [2]. They possess a huge variety of actuators and operational constraints, while persistently being subjected to disturbances. Guaranteeing a stable, reliable and efficient operation of a power system is a daunting task, while at the same time being one of the most important problems for secure power system operation [3]. Hence, it is not surprising that there exists an abundant literature on this topic dating back, at least, to the 1920s [4], [5].

Yet, in spite of these efforts, due to the complexity of the dynamics of a power system—even at the individual component level—many basic questions remain open. Therefore, typical stability analysis (and also control design) of power systems is conducted subject to several assumptions that simplify the mathematical task. Standard assumptions comprise neglecting fast dynamics [2], [6], [7] and assuming constant voltage amplitudes and small frequency variations [1, Chapter 11]. With such assumptions, it is possible to derive reduced-order synchronous generator (SG) models [1, Chapter 11] and employ algebraic models for the transmission lines [1], [2], [6]–[8], simplifying the analysis.

Unfortunately, the employed assumptions are often not physically justifiable in generic operation scenarios. In particular, the common representation of the motion of the machine rotor, i.e., the swing equation, in terms of mechanical and electrical power, instead of their corresponding torques, is an approximation which is only valid for small frequency variations around the nominal frequency [1], [2], [8]–[11].

Due to the steadily increasing penetration of fluctuating renewable energy sources, power systems worldwide often operate closer to their stability limits [12], [13]. Hence, the necessity of a more rigorous power systems stability analysis valid in large operating regions has become more compelling in the past years. In particular, it is important to derive easily and quickly verifiable analytic conditions for transient stability. This is the topic addressed in the present work.

Stability analysis of power systems employing detailed models, which are valid in a broader range of operating conditions, is a long-standing problem in the power systems literature. In this work, we consider a classical and very well known scenario called the single generator infinite bus (SGIB) model [2], [8]. Opposed to most other available analysis [1], [2], we consider a fourth-order nonlinear SG model derived from first principles. The SGIB scenario with such a model is adopted in [14], [15] where sufficient conditions for almost global asymptotic stability (GAS) are derived. The analysis in [14] proceeds along the classical lines of constructing an integro–differential equation resembling the forced pendulum equation and, subsequently, showing that the SGIB system is almost GAS if and only if the same holds for that equation. In [15] the same authors provide slightly simpler conditions for stability resulting from verifying if a real-valued nonlinear map defined on a finite interval is a contraction. But, as stated in [15], these conditions are hard to verify analytically. Furthermore, and perhaps more importantly, the geometric tools employed to establish the results in [14], [15], don’t seem to be applicable to a multi–machine power system. In [10] a scenario similar to that of the SGIB system is analyzed. However, the analysis in [10] is conducted under very stringent assumptions on the specific form of the infinite bus voltage, as well as the steady-state values of the mechanical torque.

The present paper overcomes part of the limitations in the literature by providing the following main contributions.

- Necessary and sufficient conditions for uniqueness and existence of two equilibria (modulo 2π) are established.
- Sufficient conditions for almost global attractivity of one of these equilibria are derived.
- The conservativeness of the conditions, both of them given in terms of the SGIB system parameters, is evaluated via a benchmark numerical example taken from [15].

The remainder of the paper is structured as follows. The SGIB model is introduced in Section II. The steady-state solutions of this model are investigated in Section III. To establish the attractivity result, we first construct in Section IV a new Lyapunov–like function—i.e., a function that
is not positive definite but whose derivative is negative semi–
definite. LaSalle’s invariance principle [16] then establishes
some convergence properties of bounded trajectories. The
powerful, but little known, theoretical framework of cell
structures pioneered by Leonov and co–workers [17]–[19] as
well as Noldus [20] that ensures boundedness of solutions is
then recalled in Section V. Finally, in Section VI we
give conditions on the SGIB system parameters under which
the cell structure principle is satisfied, hence completing the
almost global attractivity analysis. Section VII presents a
benchmark numerical example. The paper is concluded in
Section VIII with a summary and an outlook on future work.

It is convenient to clarify at this point two important tech-
nical issues. First, the equilibrium of the SGIB model con-
cidered in the paper cannot be rendered GAS via continuous
feedback, hence the need for the qualifier “almost”1. Indeed,
the system is naturally defined on the torus, which is not
diffeomorphic to the Euclidean space, and GAS is hampered
by a well known topological obstruction [21]. Second, as
explained above the analysis carried out in the paper does not
rely on the construction of a bona fide Lyapunov function,
which we do not prove that the equilibrium is almost GAS,
but only almost globally attractive.

II. MODEL OF A SYNCHRONOUS GENERATOR
CONNECTED TO AN INFINITE BUS

In this section the main equations and assumptions are
given for the considered SGIB system. We make some
standard assumptions on the SG [22], [23], which are also
used in [10], [14], [15]. First, the rotor is round, the machine
has one pole pair per phase, there are no damper windings
and no saturation effects as well as no Eddy currents. Second,
we assume that the rotor current i_f is a real constant. This
be achieved by choosing the excitation voltage such that
i_f is kept constant, see [10]. Third, we assume balanced
three-phase signals throughout the paper [24]. For the SG this
is equivalent to assuming a ”perfectly build” SG connected
in star with no neutral connection, as in [14], [15].

We follow the notation and modeling in [22]. Hence, we
use a generator reference direction, i.e., current flowing out
of the SG terminals is counted positively. We denote the electrical rotor angle of the SG by $\delta$ : $\mathbb{R}_{\geq 0} \to \mathbb{R}$ and
the electrical frequency by $\omega = \omega$ . Here, $\delta$ is the angle
between the axis of coil $a$ of the SG and the $d$–axis, see
[22, Figure 3.4]. For a constant rotor current $i_f$, the three-
phase electromotive force (EMF) $e_{abc} : \mathbb{R}_{\geq 0} \to \mathbb{R}^3$ is given by [22], [23]

$$e_{abc} = M_f i_f \omega \left[ \sin(\delta) \quad \sin(\delta - \frac{2\pi}{3}) \quad \sin(\delta + \frac{2\pi}{3}) \right]^T, \quad (1)$$

where $M_f \in \mathbb{R}_{\geq 0}$ is the peak mutual inductance. Likewise,
we denote the three-phase voltage at the infinite bus by

$$v_{abc} := \sqrt{2} V \left[ \sin(\delta_g) \quad \sin(\delta_g - \frac{2\pi}{3}) \quad \sin(\delta_g + \frac{2\pi}{3}) \right]^T, \quad (2)$$

1Almost GAS means that for all initial conditions, except a set of
Lebesgue measure zero, the trajectories will converge to the equilibrium.

2To establish an important result of this paper, namely convergence of
bounded solutions, it is more convenient to work with angles defined on the
real line rather than on the circle.

where $V \in \mathbb{R}_{\geq 0}$ is the root–mean–square (RMS) value of the
constant voltage amplitude (line-to-neutral) and

$$\delta_g = \delta_g(0) + \omega^* t \in \mathbb{R}, \quad (3)$$

with the grid frequency $\omega^*$ being a positive real constant.
We denote the stator resistance by $R \in \mathbb{R}_{\geq 0}$ and by $L = L_s + L_e$ the
stator inductance composed of the self–inductance $L_s \in \mathbb{R}_{\geq 0}$ and the mutual inductance $M_e \in \mathbb{R}_{< 0}$. In practice,
$L_s > -M_e$ and, hence, $L > 0$. Then the electrical

$$\text{equations describing the dynamics of the three-phase stator current } i_{abc} : \mathbb{R}_{\geq 0} \to \mathbb{R}^3 \text{ are given by}$$

$$L \frac{di_{abc}}{dt} = -R i_{abc} + e_{abc} - v_{abc}. \quad (4)$$

The SGIB model is completed with the mechanical
describing the rotor dynamics, i.e.,

$$\dot{\delta} = \omega,$$

$$J \dot{\omega} = -D \omega + T_m - T_e, \quad (5)$$

where $J \in \mathbb{R}_{> 0}$ is the total moment of inertia of the rotor
masses, $D \in \mathbb{R}_{\geq 0}$ is the damping coefficient and $T_m \in \mathbb{R}_{\geq 0}$
is the mechanical torque provided by the prime mover. Note
that we assume $T_m$ constant throughout the paper. Also, the
electrical torque $T_e$ can be written as [23]

$$T_e = \omega^{-1} i_{abc}^T e_{abc}. \quad (6)$$

For our analysis, we represent all three-phase electrical
variables in dq–coordinates with respect to the angle

$$\varphi := \omega^* t \quad (7)$$

and employ the dq–transformation matrix $T_{dq}()$ given in [1],
[2], [24]. The angle difference between the rotor angle $\delta$ and
the dq–transformation angle $\varphi$ is denoted by $\theta := \delta - \varphi$.
In dq–coordinates, the grid voltage (2) is thus given by the
costant vector (see [24]),

$$v_{dq} = \begin{bmatrix} v_d \\ v_q \end{bmatrix} = \sqrt{3} V \begin{bmatrix} \sin(\delta - \varphi) \\ \cos(\delta - \varphi) \end{bmatrix} = \sqrt{3} V \begin{bmatrix} \sin(\delta_g(0)) \\ \cos(\delta_g(0)) \end{bmatrix}, \quad (8)$$

where the second equality follows from (3). By defining $b := \sqrt{3}/2 M_f i_f$, the EMF in dq–coordinates is given by

$$e_{dq} = \begin{bmatrix} b_\omega \sin(\theta) \\ b_\omega \cos(\theta) \end{bmatrix} \quad (9)$$

By replacing the rotor angle dynamics, i.e., $\delta$, with the
relative rotor angle dynamics, i.e., $\theta$, the SGIB model given
by (4), (5) and (6) becomes in dq–coordinates

$$\dot{\theta} = \omega - \omega^*, \quad (10)$$

$$J \dot{\omega} = -D \omega + T_m - b (i_q \cos(\theta) + i_d \sin(\theta)), \quad (11)$$

$$L_i d = -R i_d - L_\omega^* i_q + b_\omega \sin(\theta) - v_d, \quad (12)$$

Here, we have used the facts that the electrical torque $T_e$ in
(6) is given in dq–coordinates by

$$T_e = \omega^{-1} i_{abc}^T e_{abc} = \omega^{-1} i_{dq}^T e_{dq} = b (i_q \cos(\theta) + i_d \sin(\theta)).$$
and that, with \( \varphi \) given in (7),
\[
\frac{d I_{dq}(\varphi)}{dt} = \omega^* \begin{bmatrix} i_q & i_d \end{bmatrix}^T,
\]
see [24, equation (4.8)]. The model (9) is used for the analysis in this paper.

Remark 1: The analysis reported in the paper can be conducted in any coordinate frame. However, we favor the one used here since it seems to be more suitable to extend the results to the multi-machine case.

III. EXISTENCE AND UNIQUENESS OF EQUILIBRIA

In this section, we investigate existence and uniqueness of equilibria of the system (9), which are denoted by \((\theta^*, \omega^*, i_d^*, i_q^*)\). To simplify the notation it is convenient to introduce two important constants
\[
c := b\sqrt{(v^2_d + v^2_q)(((\omega^*)^2 + R^2)},
\]
\[
\mathcal{P} := \frac{1}{c} \left[ -b^2\omega^* R + (T_m - D\omega^*)((\omega^*)^2 + R^2) \right].
\]
Clearly, \(c\) is nonzero if the rotor current \( i_f \) is nonzero, which is satisfied in any practical scenario.

Proposition 1: The system in (9) possesses two unique steady-state solutions (modulo \(2\pi\)) if and only if
\[
|\mathcal{P}| < 1.
\]
(12)

If and only if (12) is satisfied with equality, the system (9) has exactly one steady-state solution (modulo \(2\pi\)).

Proof: Obviously, the equilibria of the system (9) are \(2\pi\)-periodic in \(\theta\). Furthermore, we have to solve the equations
\[
\begin{align*}
\omega - \omega^* &= 0, \\
-D\omega - b_i d \sin(\theta) - b_i q \cos(\theta) + T_m &= 0, \\
-Ri_i \omega + L\omega^* i_q + b_\omega \sin(\theta) - v_i &= 0, \\
L\omega^* i_d - R_i q + b_\omega \cos(\theta) &= v_i.
\end{align*}
\]
(13)

Thus, equilibria \((\theta^*, \omega^*, i_d^*, i_q^*)\) are given by
\[
\begin{align*}
\omega^* &= \omega^*, \\
\omega^* &= \frac{b R^* \sin(\theta^*) - b L(\omega^*)^2 \cos(\theta^*) - v_i R + v_i L\omega^*}{(\omega^*)^2 + R^2}, \\
i_d^* &= \frac{b(\omega^* v_q - R_v d) \sin(\theta^*) - b(\omega^* v_d + R_v q) \cos(\theta^*)}{(\omega^*)^2 + R^2}, \\
b(\omega^* v_q - R_v d) \sin(\theta^*) - b(\omega^* v_d + R_v q) \cos(\theta^*) &= -b^2\omega^* R + (T_m - D\omega^*)((\omega^*)^2 + R^2).
\end{align*}
\]
(14)

The last equation implies that such \(\theta^*\) does exist if and only if
\[
\left| \frac{-b^2\omega^* R + (T_m - D\omega^*)((\omega^*)^2 + R^2)}{b\sqrt{(v^2_d + v^2_q)(((\omega^*)^2 + R^2)}} \right| \leq 1.
\]
(15)

Thus, condition (15) is necessary and sufficient for system (9) to have either one (equality) or exactly two (strict inequality) equilibria (modulo \(2\pi\)), completing the proof.

IV. CONVERGENCE OF BOUNDED SOLUTIONS

In this section, we derive a sufficient condition under which all bounded solutions of the system (18) converge to an equilibrium. The claim is established constructing a Lyapunov–like function and invoking LaSalle’s invariance principle [16]. Throughout the rest of the paper we make the following natural assumption.

Assumption 1: The parameters of the system (9) are such that condition (12) of Proposition 1 is satisfied and \(i_f > 0\).

Proposition 2: Consider the system (9) verifying Assumption 1 and the inequality
\[
4RD((L\omega^*)^2 + R^2) > (L\omega^*)^2.
\]
(16)

Every bounded solution tends to an equilibrium point.

Proof: Assumption 1 ensures the existence of equilibria. It is convenient to shift one of the equilibrium points to the origin via the change of coordinates
\[
\theta = \tilde{\theta} + \theta^*, \quad \omega = \tilde{\omega} + \omega^*, \quad i_d = \tilde{i}_d + i_d^*, \quad i_q = \tilde{i}_q + i_q^*.
\]

In the variables \((\tilde{\theta}, \tilde{\omega}, \tilde{i}_d, \tilde{i}_q)\) the system (9) has the form
\[
\begin{align*}
\dot{\tilde{\theta}} &= \tilde{\omega}, \\
\dot{\tilde{\omega}} &= -D(\tilde{\omega} + \omega^*) - b_i d (\tilde{i}_d + \tilde{i}_d^*) \sin(\theta^* + \tilde{\theta}) \\
&\quad - b_i q (\tilde{i}_q + \tilde{i}_q^*) \cos(\theta^* + \tilde{\theta}) + T_m, \\
\tilde{\dot{i}}_d &= -R(\tilde{i}_d^* + \tilde{i}_d^*) - Lo^*(\tilde{i}_q^* + \tilde{i}_q^*) \\
&\quad + b(\tilde{\omega} + \omega^*) \sin(\theta^* + \tilde{\theta}) - v_i, \\
\tilde{\dot{i}}_q &= -R(\tilde{i}_q^* + \tilde{i}_q^*) + Lo^*(\tilde{i}_d^* + \tilde{i}_d^*) \\
&\quad + b(\tilde{\omega} + \omega^*) \cos(\theta^* + \tilde{\theta}) - v_i.
\end{align*}
\]
(17)

Furthermore, taking into account equations (14), we get
\[
\begin{align*}
\dot{\tilde{\theta}} &= \tilde{\omega}, \\
\dot{\tilde{\omega}} &= -D\tilde{\omega} - b_i d \sin(\theta^* + \tilde{\theta}) - b_i q (\sin(\theta^* + \tilde{\theta}) - \sin(\theta^*)) \\
&\quad - b_i q \cos(\theta^* + \tilde{\theta}) - b_i q \cos(\theta^* + \tilde{\theta}) - \cos(\theta^*)), \\
\tilde{\dot{i}}_d &= -\tilde{R}_d(\tilde{i}_d^* + \tilde{i}_d^*) - Lo^* \tilde{i}_q + b_\omega \sin(\theta^* + \tilde{\theta}) - \sin(\theta^*)) \\
&\quad + b_\omega \cos(\theta^* + \tilde{\theta}), \\
\tilde{\dot{i}}_q &= -\tilde{R}_q(\tilde{i}_q + \tilde{i}_q^*) + Lo^* \tilde{i}_d + b_\omega \cos(\theta^* + \tilde{\theta}) - \cos(\theta^*)) \\
&\quad + b_\omega \cos(\theta^* + \tilde{\theta}).
\end{align*}
\]
(18)

The second step is to construct the Lyapunov–like function. Note that the electrical dynamics takes the form
\[
\tilde{L}_{dq} \begin{bmatrix} u \\
w \end{bmatrix} + \begin{bmatrix} b_\omega \sin(\theta^* + \tilde{\theta}) \\
b_\omega \cos(\theta^* + \tilde{\theta}) \end{bmatrix},
\]
(19)

where we defined
\[
\begin{bmatrix} u \\
w \end{bmatrix} := \begin{bmatrix} -R - Lo^* \\
Lo^* - R \end{bmatrix} \begin{bmatrix} \tilde{i}_d \\
\tilde{i}_q \end{bmatrix} + \begin{bmatrix} b_\omega \sin(\theta + \theta^*) - \sin(\theta^*) \\
b_\omega \cos(\theta + \theta^*) - \cos(\theta^*) \end{bmatrix}.
\]

The signs of \( c \) and, hence, \( \mathcal{P} \) defined in (11) depend on that of the constant rotor current \( i_f \). For the subsequent analysis, it is important to know the sign of \( c \), as it determines which of the two equilibria of the system (9) is stable. Yet, we remark that for \( i_f < 0 \) the analysis can be conducted in an analogous manner, see the numerical example in Section VII.
The expression above suggests the following function
\[ V(\chi) = \frac{L}{2}(u^2 + w^2) + \frac{J\omega^2}{2}[(L\omega^s)^2 + R^2] + b^2 R\omega^s(\bar{\theta} - \sin \bar{\theta}) + L(b\omega^s)^2(1 - \cos \bar{\theta}) + b\left[(L\omega^s)^2 + R^2\right]i_d^2 \left(\cos \theta^s - \cos(\theta^s + \bar{\theta}) - \bar{\theta} \sin \theta^s\right) + i_q^2 \left(\sin(\theta^s + \bar{\theta}) - \sin \theta^s - \bar{\theta} \cos \theta^s\right), \]
where the constant \( c \) is defined in (11). Together with (14), we have that
\[ b^2 R\omega^s(\bar{\theta} - \sin \bar{\theta}) + L(b\omega^s)^2(1 - \cos \bar{\theta}) + b\left[(L\omega^s)^2 + R^2\right]i_d^2 \left(\cos \theta^s - \cos(\theta^s + \bar{\theta}) - \bar{\theta} \sin \theta^s\right) + i_q^2 \left(\sin(\theta^s + \bar{\theta}) - \sin \theta^s - \bar{\theta} \cos \theta^s\right) =
\]
\[ c \int_0^\theta [\sin(\theta^s - \phi + s) - \sin(\theta^s - \phi)]ds. \]
Hence, \( V \) can be written compactly as
\[ V(\chi) = \frac{L}{2}(u^2 + w^2) + \frac{J\omega^2}{2}[(L\omega^s)^2 + R^2] +
\]
\[ + c \int_0^\theta [\sin(\theta^s - \phi + s) - \sin(\theta^s - \phi)]ds. \]
From the definition of \( V \) in (26) it follows that the function \( V \) is positive definite on the hyperplane \( \bar{\theta} = 0 \). For every integer \( k = 0, \pm 1, \pm 2, \ldots \) consider the function
\[ \hat{V}_k(\chi) = \frac{L}{2}(u^2 + w^2) + \frac{J\omega^2}{2}[(L\omega^s)^2 + R^2] - \frac{c}{2}(\bar{\theta} - 2\pi k)^2 \]
\[ + c \int_{2\pi k}^{\bar{\theta}} [\sin(\theta^s - \phi + s) - \sin(\theta^s - \phi)]ds. \]
It follows immediately that \( \hat{V}_k \) is positive definite on the hyperplane \( \bar{\theta} = 2\pi k \). Observe that the function \( \hat{V}_k \) defined in (27) differs from \( V \) defined in (23), in that the integer \( k \) appears as a scalar in the quadratic term \(-0.5c(\bar{\theta} - 2\pi k)^2\) and in the lower limit of the integral expression, which is \( 2\pi \)-periodic. Hence, the fact that by assumption \( \bar{V} \leq -\lambda \bar{V} \), i.e., condition (23), implies that also for every integer \( k \), \( \hat{V}_k \leq -\lambda \hat{V}_k \). This and the \( 2\pi \)-periodicity of the system (18) with respect to \( \bar{\theta} \), imply that for every integer \( k \) the set
\[ Z_k = \{ \chi \in \mathbb{R}^4 : \hat{V}_k(\chi) \leq 0 \} \]
is invariant with respect to solutions of the system (18).
Assume \( \chi(\cdot) \) is a solution of system (18) with initial condition \( \chi(0) = \chi_0 \). From the definition of the function \( \hat{V}_k \) in (27) we see that \( \hat{V}_k(\chi_0) \) is decreasing with respect to \( |k| \) quadratically. Hence, for any \( \chi_0 \) there exist integers \( k_1 \) and \( k_2 \), with \( k_1 < k_2 \), such that \( \hat{V}_{k_1}(\chi_0) \leq 0, \hat{\theta}(0) \geq 2\pi k_1, \) and \( V_{k_2}(\chi_0) \leq 0, \hat{\theta}(0) \leq 2\pi k_2 \). The function \( \hat{V}_{k_1} \) is positive on the plane \( \bar{\theta} = 2\pi k_1 \), and the function \( \hat{V}_{k_2} \) is positive on the plane \( \bar{\theta} = 2\pi k_2 \). Furthermore, the sets \( Z_{k_1} \) and \( Z_{k_2} \) are invariant. Consequently, we have that \( 2\pi k_1 \leq \bar{\theta}(t) \leq 2\pi k_2 \) for all \( t \geq 0 \). This completes the proof.

VI. MAIN RESULT: ALMOST GLOBAL ATTRACTIVITY

To streamline the presentation of our main result, we need the following: Given the equilibrium values \( \theta^s \) and \( \omega^s \), define the functions
\[ q(\bar{\theta}) := c \int_0^{\bar{\theta}} [\sin(\theta^s - \phi + s) - \sin(\theta^s - \phi)]ds, \]
It follows from the mean value theorem that

\[ Thus, for \( \epsilon > c \), the only solution of (32) is \( \dot{\theta}^* = 0 \) and

\[
\frac{\partial^2 h}{\partial \theta^2}\bigg|_{\theta = \dot{\theta}^*} = c \cos(\theta^* - \phi + \dot{\theta}^*) - \epsilon,
\]

which shows that for \( \epsilon > c \geq c \cos(\theta^* - \phi) \), \( \dot{\theta}^* = 0 \) is a maximum of \( h \), completing the proof.

We are now ready to state our main result.

**Theorem 1:** Consider the system (9) verifying Assumptions 1 and 2. The equilibrium point \((\theta^*, \omega^*, i^*_1, i^*_2)\) satisfying \(|\theta^* - \phi| < \frac{\pi}{2}\) (modulo \(2\pi\)) with \( \phi \) defined in (25) is locally asymptotically stable and almost globally attractive, i.e., for all initial conditions, except a set of measure zero, the solutions of the system (9) tend to that equilibrium point.

**Proof:** Assumption 1 ensures, via Proposition 1, that an equilibrium exists. To establish the local stability claim we note that from (14) and the definition of \( q(\theta) \) above, it follows that

\[
q(0) = 0, \quad q'(0) = 0, \quad q''(0) = c \cos(\theta^* - \phi)
\]

and

\[
\sin(\theta^* - \phi) = \mathcal{P},
\]

with \( \mathcal{P} \) defined in (11). From Assumption 1 we have \(|\mathcal{P}| < 1\). Therefore, the equation (33) has two roots \( \theta^* \) in the interval \([\phi, \phi + 2\pi)\). If \(|\theta^* - \phi| < \frac{\pi}{2}\), then \(q''(0) > 0\). This implies that the function \( V \) has a local minimum at the origin. Furthermore, the parameters \( \lambda \) and \( \epsilon_{\text{min}} \) only enter with negative sign in \( g(\lambda) \). Hence, Assumption 2 implies that (30) is also satisfied for \( \lambda = \epsilon_{\text{min}} = 0 \) (with \( \epsilon_{\text{min}}(\lambda)|_{(0,0)} := 0 \)) which is exactly condition (16). Thus, \( V \leq 0 \) and the zero solution of the system (18) is Lyapunov asymptotically stable (see Proposition 2). If \(|\theta^* - \phi| > \frac{\pi}{2}\), then \(q''(0) < 0\), and the zero solution of the system (18) is Lyapunov unstable.

To show almost global attractivity of the stable equilibrium, we assume in the following that the zero solution of the system (18) is Lyapunov unstable (and therefore \(|\theta^* - \phi| > \frac{\pi}{2}\)). Recall the sets \( Z_k \) defined in (28) and note that every intersection of sets \( Z_k \) is also invariant. The set \( Z_k \) is equal to \( Z_0 \) shifted in the coordinate \( \theta \) by \( 2\pi k \) to the right since \( \theta_k^* = \theta^* + 2\pi k \). Now,

\[
Z_0 = \{ \chi \in \mathbb{R}^4 : \frac{L}{2}(u^2 + w^2) + \frac{J\omega^2}{2}((\omega^s)^2 + R^2) + q(\theta) - \frac{\epsilon}{2} \omega^2 \leq 0 \}.
\]

Recall that by Lemma 1, the number \( \epsilon_{\text{min}} \) defined in (29) indeed exists. Now we check the condition of Proposition 3. To this end, we evaluate \( \frac{dV}{dt} + \lambda \dot{V} \), which yields

\[
\frac{dV}{dt} + \lambda \dot{V} = -R[u^2 + w^2] - D([(\omega^s)^2 + R^2]\omega^2 + \omega u L\omega^s \cos(\theta + \phi) - \omega w L\omega^s \sin(\theta + \phi)
\]

\[
- c\dot{\theta} \omega + \lambda\left[\frac{L}{2}(u^2 + w^2) + \frac{J\omega^2}{2}((\omega^s)^2 + R^2)\right]
\]

\[
+ c \int_{0}^{\theta} \sin(\theta^* - \phi + s) - \sin(\theta^* - \phi) ds - \frac{\epsilon}{2} \omega^2
\]

\[
\leq (\frac{L}{2})[u^2 + w^2] - \epsilon \dot{\theta} \omega - \frac{\lambda\epsilon - \epsilon_{\text{min}}}{2} \omega^2
\]

\[
- ((\omega^s)^2 + R^2)(D - \frac{J\lambda}{2}) \omega^2
\]

\[
+ \omega u L\omega^s \cos(\theta + \phi) - \omega w L\omega^s \sin(\theta + \phi)
\]

\[
= [u \ w \ \tilde{\omega} \ \tilde{\theta}] M_1 [u \ w \ \tilde{\omega} \ \tilde{\theta}]^T,
\]

with \( M_1 \) given in (37). The matrix \( M_1 \) is negative definite if and only if \( \epsilon > \epsilon_{\text{min}} \) and the matrix \( M_2 \) defined in (38) is negative definite. Similarly to the matrix \( M \) defined in (22), \( M_2 \) is negative definite if and only if \( 2R > \lambda \) and

\[ 4(R - \frac{L}{2})[(\omega^s)^2 + R^2/(D - \frac{J\lambda}{2}) - \frac{\epsilon^2}{2\lambda(\epsilon - \epsilon_{\text{min}})}] > (L\omega^s)^2. \]

In (35) the positive parameters \( \epsilon \) and \( \lambda \) have to be chosen. The maximum of the left hand side with respect to \( \epsilon \) is attained at \( \epsilon = 2\epsilon_{\text{min}} \). For this choice, (35) takes the form

\[
4(R - \frac{L}{2})[(\omega^s)^2 + R^2/(D - \frac{J\lambda}{2}) - \frac{2\epsilon_{\text{min}}\lambda}{J}] > (L\omega^s)^2.
\]

Consider the following polynomial

\[
f(\lambda) := ((\omega^s)^2 + R^2/(D - \frac{J\lambda^2}{2}) - 2\epsilon_{\text{min}},
\]

and denote by \( \lambda_1 \) its smallest root, that is,

\[
\lambda_1 = D - \sqrt{D^2 - \frac{4\epsilon_{\text{min}}}{(L\omega^s)^2 + R^2}}.
\]

If \( \lambda_1 < \frac{2L}{\lambda} \), then on the interval \([\lambda_1, \frac{2L}{\lambda}] \) there is a unique
almost globally attractive equilibrium. This result coincides with Proposition 2 ensuring that the inequalities (30) of Assumption 2 are satisfied with condition (16) of Proposition 2 is satisfied. Consequently, all solutions \( \theta, \omega, i_d, i_q \) of the system (9) are bounded and tend to an equilibrium point. Recall that one of the two equilibria of the system (9) (modulo \( 2\pi \)) is stable and the other one is unstable. Thus, for all initial conditions, except a set of measure zero, the solutions of the system (9) tend to the stable equilibrium point. This shows that the latter is almost globally attractive and completes the proof. 

**Remark 2:** Note that if \( P = 0 \) (and therefore \( |\theta^* - \phi| = \frac{\pi}{2} \)) then \( \epsilon_{\text{min}} = 0 \), and the inequality (36) is equivalent to (16).

**Remark 3:** The related analysis in [10] critically relies on imposing a specific value for the mechanical torque \( T_m \) and on the knowledge of the stationary rotor currents \( i^s \). Such restrictions do not apply in the present case.

**VII. NUMERICAL EXAMPLE**

We investigate the effectiveness of the inequalities (30) of Assumption 2 via a numerical benchmark example directly from [15]. Note that in the example of [15] the rotor current \( i_q < 0 \). Thus, \( b < 0 \) and \( c < 0 \), see (11). In our notation, this corresponds to the (potentially) stable equilibrium being shifted by \( \pi \). Indeed, conditions (11), (12) and (16) are satisfied for this example. Hence, the system (9) has two equilibria and the proof of Theorem 1 implies that the equilibrium with \( |\theta^* - \phi| > \frac{\pi}{2} \) is locally asymptotically stable. In addition, inequalities (30) of Assumption 2 are satisfied with \( \epsilon_{\text{min}} = 82.12 \) and \( \lambda = 23.81 \). Consequently, by Theorem 1, the equilibrium with \( |\theta^* - \phi| > \frac{\pi}{2} \) is also an almost globally attractive equilibrium. This result coincides with the conclusions in [15].

**VIII. CONCLUSIONS**

A complete stability analysis of a realistic SGIB model has been presented. First, it is shown that (12)—with \( P \) defined in (11)—is a necessary and sufficient condition for existence of equilibria. Then, it is proven that if the inequalities (30) of Assumption 2 hold then almost all trajectories converge to a stable equilibrium point. The conservativeness of the estimates have been assessed via a numerical benchmark problem.

The main topic of future research is the extension of these results to the multi-machine case. Given the “scalable” nature of the analysis tools employed here this seems a feasible—albeit difficult—task.

**REFERENCES**


\[
M_1 = \begin{bmatrix}
-R + \frac{L_\lambda}{2} & 0 & 0 & \frac{L_\omega^* \cos(\theta^* + \theta')}{2}
\end{bmatrix}
\]

\[
M_2 = \begin{bmatrix}
-R + \frac{L_\lambda}{2} & 0 & 0 & \frac{L_\omega^* \cos(\theta^* + \theta')}{2}
\end{bmatrix}
\]