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On the Assignment Problem with a Nearly Monge Matrix and its Applications in Scheduling

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1 Introduction

The linear assignment problem is one of the most extensively studied problems in combinatorial optimization. While it is generally solvable in $O(n^3)$ time, there are a number of important special cases which can be solved faster, see [1]. The famous class of Monge cost matrices is of particular importance from both theoretical and applied viewpoints. For an $n \times n$ Monge matrix $W = (w_{ij})$, every quadruple of entries in rows i, s $(i < s)$ and columns j, t $(j < t)$ satisfies

$$w_{ij} + w_{st} \leq w_{it} + w_{sj}. \tag{1}$$

This property is associated with the French mathematician Gaspard Monge who in 1781 made the observation that efficient transportation from supply points i, s to demand points j, t should use the arcs that do not intercross. Since the greedy algorithm solves the transportation problem with a Monge matrix, an optimal solution to the assignment problem with a Monge matrix has a diagonal structure. This holds even in the multi-dimensional case, with a generalized Monge condition. For surveys of results on Monge matrices see [2, 3].

Cost matrices satisfying the Monge conditions arise in many applications. Some of them give rise to ∞-entries which model forbidden assignments: if $w_{ij} = \infty$, then i cannot be assigned to j. Depending on the position of the ∞-entries, the Monge property may still be satisfied, so that the problem with ∞'s remain greedily solvable. The generalized version of a multi-dimensional problem of such type is studied in [7] which exploits the relationship between multi-dimensional Monge arrays (with and without ∞'s) and submodular functions. Note that the requirement of [7] that the finite entries form a sublattice implies that ∞'s do respect the Monge property.

In general, however, an arbitrary introduction of ∞-entries may destroy the Monge property in a matrix that initially satisfied it. We call an $n \times n$ matrix W *nearly Monge* if the Monge condition holds for all quadruples with finite entries, while it may be violated for quadruples containing ∞-entries. Two typical scenarios that give rise to assignment problems with nearly Monge matrices are related to satellite communication (SC) and synchronous open shop scheduling (SO).

In the basic SC problem, m senders should transmit messages to n receivers. The duration of a transmission from sender s, $1 \leq s \leq m$, to receiver r, $1 \leq r \leq n$, is t_{sr}.

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Each sender and each receiver can handle at most one message at a time. In order to avoid conflicts, configuration changes are done simultaneously, so that a feasible solution consists of periods in which simultaneous transmissions happen. The duration of every period is given by the time needed to send the longest message. The goal is to minimize the total duration of completing all transmissions. Various versions of this problem were extensively studied since the 80s, see [5, 6, 8, 9]. The version with two senders ($m = 2$) corresponds to the two-dimensional assignment problem with weights

$$w_{ij} = \max\{t_{1i}, t_{2j}\}. \tag{2}$$

Here w_{ij} is the the cost of performing two transmissions $1 \to i$ and $2 \to j$ simultaneously. Notice that if the two arrays (t_{1i}) and (t_{2j}) are renumbered in non-decreasing order, then $W = (w_{ij})$ is a Monge matrix. Additional condition $w_{ij} = \infty$ for $i = j$ prohibits simultaneous transmission to the same receiver and makes the resulting cost matrix nearly Monge. The general version of the SC problem with $m > 2$ senders corresponds to the m-dimensional assignment problem with costs satisfying the multi-dimensional nearly Monge condition.

The SO scheduling problem can be seen as a reformulation of the SC problem in scheduling terminology. There are given m machines and n jobs, together with operation durations t_{sr}, where s is the machine index, $1 \le s \le m$, and r is the job index, $1 \le r \le n$. At any time, a machine can process at most one job and a job can be processed by at most one machine. In the synchronous version of open shop, job changes on machines should be done simultaneously, so that a schedule consists of periods with changes happening at the end of every period. If two jobs i and j are to be processed within the same period by machines 1 and 2, then the duration of the period w_{ij} is defined by (2). The objective is to schedule all operations so that the makespan is minimum. Similar to the SC problem, the SO problem can be modelled as the m-dimensional assignment problem with a nearly Monge cost matrix, where ∞-entries prohibit allocation of two operations of the same job to one period. We are not aware of any results for the SO problem; its counterpart of the flow shop type was studied in [10, 11].

Both problems, SC and SO, have a common underlying model known as *max-weight edge coloring* (MEC). The MEC problem has been extensively studied since the 2000s, see, e.g., [4]. Given a weighted graph G and a feasible k-edge-coloring f, the weight of a color $c \in \{1, \ldots, k\}$ is defined as $w(c) = \max\{w_e \mid f(e) = c\}$. The goal is to find a feasible edge-coloring of minimum total weight $\sum_{c=1}^{k} w(c)$. The SO and SC problems correspond to the MEC problem defined on a complete bipartite graph $G = K_{m,n}$ with a weight matrix of type (2).

Observe that the subject of our study is a generalization of the problems mentioned above, namely the assignment problem with a nearly Monge matrix, which entries are not necessarily defined by (2). We denote such a problem by $A(W, d, \lambda)$, where W is a d-dimensional nearly Monge cost matrix with at most λ incompatible partners for a fixed index $i_u = i_u^*$ in any position i_v, $v \neq u$.

2 Structural properties, algorithms and complexity

The NP-hardness of the problem $A(W, d, \lambda)$ with an arbitrary d follows from a similar result known for the MEC problem [8].

Theorem 1 *Problem $A(W, d, \lambda)$ is strongly NP-hard for an arbitrary d, even if $\lambda = 1$ and even if there are only three different finite weights in W.*

The main subject of our study is problem $A(W, d, \lambda)$ with a fixed d, which is a typical condition for applied problems. For this problem we establish a so-called "corridor property" that characterizes the structure of an optimal solution. It implies that 1-entries of a solution matrix belong to a corridor of limited width around the main diagonal.

Theorem 2 *For problem $A(W, d, \lambda)$ there exists an optimal solution such that its* 1*-entries belong to a corridor of width* $2(d-1)\lambda$ *around the main diagonal.*

Theorem 2 gives rise to an efficient dynamic programming algorithm that solves problem $A(W, d, \lambda)$ in $O(n)$ time, if d and λ are fixed, and in FPT time, if d and λ are parameters. Since the three problems discussed in the Introduction are special cases of problem $A(W, d, \lambda)$ with $\lambda = 1$ after sorting is performed in order to achieve the Monge condition for finite entries, our result has the following implications.

Corollary 1 *Problems (1) MEC in a complete bipartite graph $K_{m,n}$, (2) SC with m senders and n receivers and (3) SO with m machines and n jobs are solvable in $O(n \log n)$ time if m (or symmetrically n) is fixed, and in FPT time, if m (or symmetrically n) is viewed as a parameter.*

References

[1] R. E. Burkard, M. Dell'Amico, S. Martello (2009). Assignment Problems. SIAM, Philadelphia, USA.

[2] R. E. Burkard, B. Klinz, R. Rudolf (1996). Perspectives of Monge properties in optimization. Discr. Appl. Math. 70, 95–161.

[3] R. E. Burkard (2007). Monge properties, discrete convexity and applications. Eur. J. of Oper. Res. 176, 1–14.

[4] M. Demange, B. Escoffier, G. Lucarelli, I. Milis, J. Monnot, V. Th. Paschos, D. de Werra (2008). Weighted Edge Coloring. In Combinatorial Optimization and Theoretical Computer Science (ed V. Th. Paschos), ISTE, London.

[5] I. S. Gopal, C. K. Wong (1985). Minimizing the number of switchings in an SS/TDMA system. IEEE T. Commun. 33, 497–501.

[6] A. Kesselman, K. Kogan (2007). Nonpreemtive scheduling of optical switches. IEEE T. Commun. 55, 1212–1219.

[7] M. Queyranne, F. Spieksma, F. Tardella (1998) A general class of greedily solvable linear programs. Math. Oper. Res. 23, 892–908.

[8] F. Rendl (1985). On the complexity of decomposing matrices arising in satellite communication. Oper. Res. Lett. 4, 5–8.

[9] C. C. Ribeiro, M. Minoux, M. C. Penna (1989). An optimal column-generation-with-ranking algorithm for very large scale set partitioning problems in traffic assignment. Eur. J. of Oper. Res. 41, 232–239.

[10] B. Soylu, Ö. Kirca, M. Azizoğlu (2007). Flow shop-sequencing problem with synchronous transfers and makespan minimization. Int. J. of Prod. Res. 45, 3311–3331.

[11] S. Waldherr, S. Knust (2015). Complexity results for flow shop problems with synchronous movement. Eur. J. of Oper. Res. 242, 34–44.