Nash Dominance with Applications to Equilibrium Problems with Equilibrium Constraints

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Abstract We examine a novel idea for the detection of Nash Equilibrium developed in [14] and apply it to Equilibrium Problems with Equilibrium Constraints (EPECs). EPECs are Nash games which uniquely feature players constrained by a condition governing equilibrium of a parametric system. By redefining the selection criteria used in evolutionary methods, EPECs can be solved using Evolutionary Multiobjective Optimization algorithms. We give a proposed algorithm (NDEMO) and illustrate it with numerical examples.

1 Introduction

A major trend in the provision of transportation services and facilities has been deregulation coupled with the private sector playing a larger role. When it occurs in highway [30] or transit [31], entities providing such services face competition from others with similar offerings. It is of interest to regulators to understand how such organizations make decisions on their service levels in this deregulated environment.

In this environment, the service levels provided are an outcome of a non-cooperative Nash game [17] amongst the players. However in transportation, this game possesses a feature that distinguishes it from the classic Nash game: The players' actions are constrained by a condition defining equilibrium in the transportation system [5]. In particular, the route choice decisions of users of a transportation network, which satisfy Wardrop's Equilibrium Principle [28], are parameterized in the decision variables of these firms. Therefore this is a hierarchical (i.e. leader-follower) game with the firms as leaders at the upper level engaged in a Nash game and travelers as followers at the lower level obeying an equilibrium condition. Thus, in this context, the terms "firms", "leaders" and "players" are synonymous.

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The game described in the foregoing is an instance of a broader class of Equilibrium Problems with Equilibrium Constraints (or EPECs) ([15],[16]). EPECs have emerged as an area of research ([1],[27]) in mathematics applicable to transportation systems management and other disciplines ([8],[16]). This paper focuses on the determination of strategic variables for each profit maximizing leader when in competition with others. To avoid cumbersome notation, we implicitly assume henceforth that the equilibrium of the system acts as a binding constraint on the leaders.

In this paper, we are concerned with games where the payoffs to the players are continuous and the strategic decision variables are subsets of the real line (see e.g. Chapter 6 of [29]). Much of the game theory literature deals with games that are either zero sum (e.g. Tic-Tac-Toe [29]) or where the actions of players are constrained to be in a discrete set (e.g. Prisoner's Dilemma [29]) and thus solution algorithms proposed for these are generally not applicable to EPECs. Our contribution is in the application of an evolutionary algorithm based on the proximity of solutions to a Nash Equilibrium (NE) to the EPEC.

This paper is organized as follows. In the next section, the notions associated with the Nash game underlying the behavior of the leaders in the EPEC are introduced. Section 3 reviews deterministic (i.e. gradient based) and evolutionary approaches for computing NE in EPECs. Section 4 elucidates the Nash Domination criteria developed in [14] and provides an algorithm. Section 5 presents numerical examples of the solution of EPECs utilizing the concept of Nash Domination. Section 6 concludes the paper with a summary and directions for further research.

2 Nash Equilibrium

The leaders' problem in the EPEC is a single shot normal form game with a set of N players indexed by $i \in \{1,2,\ldots,n\}$ and each player can play a strategy $s_i \in S_i$ which all players are assumed to announce simultaneously. $S = \prod_{i=1}^{n} S_i$ is the collective action space for all players. It is convenient to denote s_{-i} as the combined strategies of all players in the game excluding that of player i i.e. $s_{-i} \equiv (s_1,\ldots,s_{(i-1)},s_{(i+1)},\ldots,s_n)$. So we have that $s \equiv (s_i,s_{-i})$ and we call s a strategy profile of all players in the game. Let $U_i(s)$ be the payoff to player $i, i \in N$ if s is played. Then a combined strategy profile $s^* = (s_1^*,s_2^*,\ldots,s_n^*) \in S$ is a Nash Equilibrium for the game if the following holds:

$$U_i(s_i^*,s_{-i}^*) \geq U_i(s_i,s_{-i}^*) \ \forall s_i \in S_i \ ,\forall i \in \{1,2,\ldots,N\} \tag{1}$$

At a Nash Equilibrium no player can benefit (increase individual payoffs) by unilaterally deviating from her current strategy. As players do not cooperate, each player is doing the best she can taking into account what her competitors are doing [6]. Henceforth we refer to the NE problem as the determination of strategies that satisfy Equation 1.

3 Computation of Nash Equilibrium

3.1 Deterministic Approaches

In a game, the optimal move for a player is governed by her best response function. If $U_i(s)$ is continuously differentiable, then the best response function for player i is given by $\frac{dU_i(s_i,s_{-i})}{ds_i}$ ([6], [29]). The NE is the intersections of these best response functions for all players which amounts to finding solutions to N simultaneous equations i.e. solving $\frac{dU_i(s_i,s_{-i})}{ds_i} = 0, \forall i \in \{1,2,...,N\}$.

While useful for providing insights into the behavior of players, the analytical method is not feasible for realistic problems and even less so for EPECs due to the binding equilibrium condition. Thus the practical approach for finding NE is by using variants of fixed point iteration (e.g. nonlinear Gauss-Siedel) ([9],[27]) or by formulating it as a Complementarity Problem [10]. Applications of these methods are found in (e.g. [7], [13]). Convergence of these algorithms rely on the payoff functions being continuously differentiable and possessing diagonally dominant Jacobians ([6], Theorem 4.1, pp. 280). However, if the payoff functions of the players are not concave, there may exist NE that satisfy Equation 1 locally but not globally. This is known as a "local NE trap" ([26], Definition 3, pp.306). There is thus a parallel with the literature on multimodal function optimization where the potential for multiple optima cannot be ignored. Thus apart from their differentiability requirements, another drawback of deterministic approaches is that they can fall prey to the local NE trap, an occurrence crucially dependent on the starting point used in these algorithms. For details of these and other deterministic methods, see [5].

3.2 Evolutionary Methods

Due to the ability of evolutionary algorithms in dealing with non-smooth and non-differentiable functions and their reported success in escaping local optima and potentially a local NE trap, evolutionary counterparts of deterministic fixed point iteration methods were proposed in ([22],[23],[25]).

Another strand of research has been the exploitation of coevolution since it was first demonstrated in tackling multi-dimensional function optimization (e.g. [19]). Several subpopulations (one representing each problem dimension) are evolved simultaneously to avoid premature convergence and to widen the search of the problem space. Ideas from coevolution have been exported into algorithms designed for the detection of NE; here each subpopulation encodes the strategies of individual players ([3],[18],[21]). However doubts have been cast on the performance of coevolutionary methods. In [26], the coevolutionary algorithm had to be hybridized with local search techniques to enable successful detection of NE. [11] developed a coevolutionary particle swarm optimization method which attempted to detect the NE by learning the best response functions of the players. Instead of using the coevolutionary paradigm of previous works, a novel idea exploiting the concept of Nash Dominance was proposed [14] to find NE as discussed in Section 4.

4 Nash Domination

At their most abstract level, evolutionary multiobjective (EMO) algorithms ([2],[4]) apply stochastic operators to a parent population with the aim of evolving a fitter child population to solve vector valued optimization problems. Subsequently, in the selection phase, a comparison is made between a chromosome x from the parent population and a chromosome y from the child population on the basis of fitness and the weaker of the two is discarded. Given that one of the tasks in EMO is to identify the entire Pareto frontier [4], fitness is assigned based on Pareto Domination (PD): x Pareto Dominates y if x is strictly no worse off than y in all objectives *and* x is better than y in at least one objective ([4], Definition 2.5, pp. 28).

[14] define a concept analogous to PD called Nash Domination for the NE problem. A chromosome here represents the strategies of all N players concatenated into a vector i.e. a strategy profile. Then instead of using PD to compare two chromosomes i.e. two strategy profiles, Nash Domination operates by counting the number of players that can benefit if each player switches strategies *in turn*. The *fewer* the number of players that can benefit by deviating from one profile compared to the other, the closer the former is to a NE as defined in Section 2.

Consider two strategy profiles $\{a,b\} \in S, (a \equiv (a_1,\ldots,a_n), b \equiv (b_1,\ldots,b_n))$, and define an operator $k : S \times S \to N$ associating the cardinality of a set defined by 2:

$$\{i \in \{1,\ldots,n\} \,|U_i(b_i,a_{-i}) \geq U_i(a), b_i \neq a_i\} \tag{2}$$

This set defined by (2) comprises the players that would benefit by playing b_i when everyone else plays a_{-i}. The total number of players in this set is given by $k(a,b)$. A similar interpretation applies, *mutatis mutandis*, for $k(b,a)$. Note that to evaluate $k(a,b)$ and $k(b,a)$, the payoff to each player, individually, from deviating has to be computed. Then in a pairwise comparison of two strategy profiles, either one of the following must be true: ([14], Remark 4, pp. 365)

1. $k(a,b) < k(b,a) \to a$ Nash Dominates b or
2. $k(b,a) < k(a,b) \to b$ Nash Dominates a or
3. $k(a,b) = k(b,a) \to a$ and b are Nash Non Dominated (NND) with respect to each other.

From the proof ([14], Proposition 9, pp. 366) that all NND chromosomes are NE, all that is needed is to apply this selection criteria to check for Nash Domination instead of PD via an EMO algorithm. The method would then converge to the NE instead of locating the Pareto Front. A proposed Nash Domination Evolutionary Multiplayer Optimization (NDEMO) algorithm is given in Algorithm 1. NDEMO is based on the method of [24] which relies on Differential Evolution (DE) [20]. Note that any other EMO algorithm (see e.g. [2], [4] for options) can be used.

Algorithm 1 :Nash Domination Evolutionary Multiplayer Optimization (NDEMO)

Input: h, Max_{it}, ε, DE Control Parameters, payoff functions
$it \leftarrow 0$
Randomly initialize parent strategy profiles $\mathscr{P}$
Evaluate payoffs to players with $\mathscr{P}$
Check convergence $\mathscr{P}$
while $it < Max_{it}$ or $\mathscr{P}$ not converged **do**
  Apply DE operators to create child strategy profiles $\mathscr{C}$:
  $\mathscr{C} \overset{DE}{\leftarrow} \mathscr{P}$
  Evaluate payoffs to players with $\mathscr{C}$
  Perform Pairwise Nash Domination Comparison between $\mathscr{P}$ and $\mathscr{C}$:
  for $j = 1$ to h **do**
    $a \leftarrow \mathscr{P}_j^{it}$
    $b \leftarrow \mathscr{C}_j^{it}$
    if $k(a,b) < k(b,a)$ **then**
      reject b
      $\mathscr{R} \leftarrow a$
    else if $k(b,a) < k(a,b)$ **then**
      reject a
      $\mathscr{R} \leftarrow b$
    else
      $\mathscr{R} \leftarrow a$
      $\mathscr{R} \leftarrow b$
    end if
  end for
  if size of $\mathscr{R} > h$ **then**
    Randomly cull $\mathscr{R}$ until h remain
    $\mathscr{P}^{(it+1)} \leftarrow \mathscr{R}$
  end if
  Convergence Check:
  Randomly choose a chromosome (t) from $\mathscr{P}^{it+1}$
  Compute norm between t and every other member in $\mathscr{P}^{it+1}$
  if norm $\leq \varepsilon$ **then**
    Terminate
  end if
  $it \leftarrow it + 1$
end while
Output: Nash Non Dominated Solutions

NDEMO works as follows: The user specifies the maximum number of generations Max_{it}, the population size h, the convergence criteria, $\varepsilon(> 0)$, control parameters required in DE [20] and a procedure to compute payoffs. Initial parent strategy profiles $\mathscr{P}$ are generated randomly. Then child strategy profiles $\mathscr{C}$ are created by applying the DE operators via the stochastic combination of randomly chosen parents as discussed in [20]. At each generation, parent and child strategy profiles are compared one by one pairwise, following the Nash Domination procedure described previously. Those that are NND can potentially be parents for the next generation so if the number of NND chromosomes exceeds h, we randomly cull them ($\mathscr{R}$) so that there will always be only h parents. The algorithm is then repeated until either

Max_{it} is reached or when convergence (measured by the euclidean norm between a randomly chosen chromosome t and the rest of $\mathscr{P}$) is achieved.

5 Numerical Examples

The examples presented are typical of situations when a private profit maximizing firm competes with others in the operation of private roads. The interaction between these firms and users of the highway network is depicted in Figure 1. To explicitly account for the hierarchical nature of the game and that route choices of users on the network must satisfy Wardrop's Equilibrium Condition [28], a traffic assignment problem is solved for a given strategy profile of the players to obtain the traffic flows which each player's payoffs depend on.

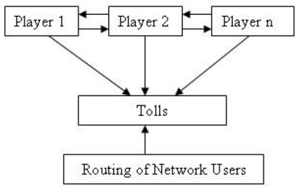

Fig. 1 Hierarchical Game with Equilibrium Route Choice

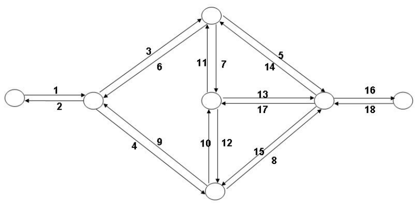

Fig. 2 Highway Network with 18 one way roads from [12] (links labeled are road numbers)

This example is taken from [12]. There are two players and each firm chooses toll levels (one firm per road) on the network (see Figure 2) to maximize toll revenues (given as the product of tolls and traffic flows). Assume firstly that roads 7 and 10 are the only tolled roads. This example was solved as a complementarity problem in [12] and via a Coevolutionary Particle Swarm Algorithm in [11]. We used a population size of 20 chromosomes, DE control parameters from [24] and terminate the algorithm when $\varepsilon \leq 1e-4$. NDEMO took 38 minutes to converge to the tolls (in seconds) shown in Table 1 which agrees with previous results.

Next we consider the situation when, in addition to roads 7 and 10 being tolled, another player maximizes profits by charging tolls on road 17. The results are reported in Table 2. Although NDEMO again successfully converged to the NE (as verified by solving it as a complementarity problem following the method described in [12]), this time NDEMO took 54 minutes to meet the same convergence criteria. Thus with one additional player, the time taken has increased by 42% over the 2 player case. The increase in computing time stems from the domination checking procedure combined with the hierarchical nature of the game. This translates into the requirement to calculate the profits to each firm from deviating (so as to obtain $k(a,b)$ and $k(b,a)$) by solving a traffic assignment problem for each player in turn.

Table 1 Tolls (seconds) in Two Player Model

Firm	Road	NDEMO	[11]	[12]
1	7	141.36	141.36	141.37
2	10	138.28	138.29	138.29

Table 2 Tolls (seconds) in Three Player Model

Firm	Road	NDEMO	Method of [12]
1	7	140.94	140.96
2	10	137.51	137.56
3	17	711.25	712.88

6 Conclusions

In this paper, we proposed modifying an EMO algorithm for solving EPECs by extending the procedure suggested in [14]. This revised algorithm (NDEMO) enabled us to handle Nash games where players encounter a system equilibrium constraint. Numerical examples illustrating competition in private sector provision of highway transportation were given to demonstrate the performance of the proposed algorithm. While the examples suggest that this could be a potentially useful method for EPECs, we stress the need, in the pairwise comparison, to compute the payoff to each player, one by one, from deviating. This implies that the computational complexity of NDEMO increases significantly as the number of players increase as evidenced by the increase in computational times required in our examples.

An area of further research would be the effects of the control parameters of NDEMO on the speed of convergence to NND solutions since we have used parameters suggested in [24]. NDEMO could also be extended to other domains where EPECs are applicable such as electricity markets (e.g. [8]). Additionally, comparisons with other algorithms are currently being undertaken.

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