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AN EXAMPLE OF AN INFINITE SET OF ASSOCIATED PRIMES OF A LOCAL COHOMOLOGY MODULE

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0. Introduction

Let \((R, m)\) be a local Noetherian ring, let \(I \subset R\) be any ideal and let \(M\) be a finitely generated \(R\)-module. It has been long conjectured that the local cohomology modules \(H^i_I(M)\) have finitely many associated primes for all \(i\) (see Conjecture 5.1 in [H] and [L].)

If \(R\) is not required to be local these sets of associated primes may be infinite, as shown by Anurag Singh in [S], where he constructed an example of a local cohomology module of a finitely generated module over a finitely generated \(\mathbb{Z}\)-algebra with infinitely many associated primes. This local cohomology module has \(p\)-torsion for all primes \(p \in \mathbb{Z}\).

However, the question of the finiteness of the set of associated primes of local cohomology modules defined over local rings and over \(k\)-algebras (where \(k\) is a field) has remained open until now. In this paper I settle this question by constructing a local cohomology module of a local finitely generated \(k\)-algebra with an infinite set of associated primes, and I do this for any field \(k\).

1. The example

Let \(k\) be any field, let \(R_0 = k[x, y, s, t]\) and let \(S = R_0[u, v]\). Define a grading on \(S\) by declaring \(\deg(x) = \deg(y) = \deg(s) = \deg(t) = 0\) and \(\deg(u) = \deg(v) = 1\). Let \(f = sx^2v^2 - (t+s)xyuv + ty^2u^2\) and let \(R = S/fS\). Notice that \(f\) is homogeneous and hence \(R\) is graded. Let \(S_+\) be the ideal of \(S\) generated by \(u\) and \(v\) and let \(R_+\) be the ideal of \(R\) generated by the images of \(u\) and \(v\).

Consider the local cohomology module \(H^2_{R_+}(R)\): it is homogeneously isomorphic to \(H^2_{S_+}(S/fS)\) and we can use the exact sequence

\[
H^2_{S_+}(S)(-2) \xrightarrow{f} H^2_{S_+}(S) \longrightarrow H^2_{S_+}(S/fS) \longrightarrow 0
\]

of graded \(R\)-modules and homogeneous homomorphisms (induced from the exact sequence

\[
0 \longrightarrow S(-2) \xrightarrow{f} S \longrightarrow S/fS \longrightarrow 0
\]

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to study $H^2_{R_+}(R)$. Furthermore, we can realize $H^2_{S_+}(S)$ as the module $R_0[u^-, v^-]$ of inverse polynomials described in [3S, 12.4.1]: this graded $S$-module vanishes beyond degree $-2$, and, for each $d \geq 2$, its $(−d)$-th component is a free $R_0$-module of rank $d - 1$ with base $(u^{-α}v^{-β})_{α, β > 0, α + β = −d}$. We will study the graded components of $H^2_{S_+}(S/f S)$ by considering the cokernels of the $R_0$-homomorphisms

$$f_{−d} : R_0[u^−, v^−]_{−d−2} \longrightarrow R_0[u^−, v^−]_{−d} \quad (d \geq 2)$$

given by multiplication by $f$. In order to represent these $R_0$-homomorphisms between free $R_0$-modules by matrices, we specify an ordering for each of the above-mentioned bases by declaring that

$$u^{α_1}v^{β_1} < u^{α_2}v^{β_2}$$

(where $α_1, β_1, α_2, β_2 < 0$ and $α_1 + β_1 = α_2 + β_2$) precisely when $α_1 > α_2$. If we use this ordering for both the source and target of each $f_d$, we can see that each $f_d$ ($d \geq 2$) is given by multiplication on the left by the tridiagonal $d - 1$ by $d + 1$ matrix

$$A_{d−1} := \begin{pmatrix}
    sx^2 & −xy(t + s) & ty^2 & 0 & \ldots & 0 \\
    0 & sx^2 & −xy(t + s) & ty^2 & 0 & \ldots & 0 \\
    0 & 0 & sx^2 & −xy(t + s) & ty^2 & \ldots & 0 \\
    \vdots \\
    0 & \ldots & sx^2 & −xy(t + s) & ty^2
\end{pmatrix}.$$ 

We also define

$$\overline{A}_{d−1} := \begin{pmatrix}
    s & -(t + s) & t & 0 & \ldots & 0 \\
    0 & s & -(t + s) & t & 0 & \ldots & 0 \\
    0 & 0 & s & -(t + s) & t & \ldots & 0 \\
    \vdots \\
    0 & \ldots & s & -(t + s) & t
\end{pmatrix}$$

obtained by substituting $x = y = 1$ in $A_{d−1}$.

Let also $τ_i = (−1)^i(t^i + st^{i−1} + \cdots + s^{i−1}t + s^i)$.

1.1. Lemma.

(i) Let $B_i$ be the submatrix of $\overline{A}_i$ obtained by deleting its first and last columns. Then $\det B_i = \tau_i$ for all $i \geq 1$.

(ii) Let $S$ be an infinite set of positive integers. Suppose that either $k$ has characteristic zero or that $k$ has prime characteristic $p$ and $S$ contains infinitely many integers of the form $p^n - 2$. The $(k[s,t])$-irreducible factors of $\{τ_i\}_{i \in S}$ form an infinite set.
\textbf{Proof.} We prove the first statement by induction on \(i\). Since

\[
\det B_1 = \det(-t - s) = -t - s \quad \text{and} \quad \det B_2 = \det\begin{pmatrix} -t - s & t \\ s & -t - s \end{pmatrix} = t^2 + st + s^2,
\]

the lemma holds for \(i = 1\) and \(i = 2\). Assume now that \(i \geq 3\). Expanding the determinant of \(B_i\) by its first row and applying the induction hypothesis we obtain

\[
\det B_i = (-t - s) \det B_{i-1} - st \det B_{i-2}
\]

\[
= (-1)^{i-1}(-t - s)(t^{i-1} + \cdots + s^{i-2}t + s^{i-1}) - (-1)^{i-2}st(t^{i-2} + \cdots + s^{i-3}t + s^{i-2})
\]

\[
= (-1)^i\left[(t^i + \cdots + s^{i-2}t^2 + s^{i-1}t) + (st^{i-1} + \cdots + s^{i-1}t + s^i) - (st^{i-1} + \cdots + s^{i-2}t^2 + s^{i-1}t)\right]
\]

\[
= (-1)^i(t^i + st^{i-1} + \cdots + s^{i-1}t + s^i).
\]

We now prove the second statement. Define \(\sigma_i = t^i + t^{i-1} + \cdots + t + 1\) and notice that it is enough to show that the set of irreducible factors of \(\{\sigma_i\}_{i \in S}\) is infinite. Let \(\mathcal{I}\) be the set of irreducible factors of \(\{\sigma_i\}_{i \in S}\). If \(k\) has characteristic zero consider \(\mathbb{Q}[\mathcal{I}] \supseteq \mathbb{Q}\), the splitting field of this set of irreducible factors. If \(\mathcal{I}\) is finite, \(\mathbb{Q}[\mathcal{I}] \supseteq \mathbb{Q}\) is finite extension which contains all \(i\)th roots of unity for all \(i \in S\), which is impossible.

Assume now that \(k\) has prime characteristic \(p\). Let \(F\) be the algebraic closure of the prime field of \(k\). For any positive integer \(m\)

\[
\frac{d}{dt}(t^{p^m-1} - 1) = -1
\]

so \(\sigma_{p^m-2} = (p^m-1)/(t-1)\) has \(p^m - 2\) distinct roots in \(F\) and, therefore, the roots of \(\{\sigma_s\}_{s \in S}\) form an infinite set.

\[
\square
\]

1.2. \textbf{Theorem.} \textit{For every} \(d \geq 2\) \textit{the} \(R_0\)-\textit{module} \(H^2_{R_+}(R)_{-d}\) \textit{has} \(\tau_{d-1}\)-\textit{torsion}. \textit{Hence} \(H^2_{R_+}(R)\) \textit{has infinitely many associated primes.}

\textbf{Proof.} For the purpose of this proof we introduce a bigrading in \(R_0\) by declaring \(\deg(x) = (1,0)\), \(\deg(y) = (1,1)\) and \(\deg(t) = \deg(s) = (0,0)\).

We also introduce a bigrading on the free \(R_0\)-modules \(R^n_0\) by declaring \(\deg(x^\alpha y^\beta s^a t^b e_j) = (\alpha + \beta, \beta + j)\) for all non-negative integers \(\alpha, \beta, a, b\) and all \(1 \leq j \leq n\). Notice that \(R^n_0\) is a bigraded \(R_0\)-module when \(R_0\) is equipped with the bigrading mentioned above.

Consider the \(R_0\)-module \(\text{Coker} A_{d-1}\); the columns of \(A_{d-1}\) are bihomogeneous of bidegrees

\[(2,1), (2,2), \ldots, (2, d+1)\]

We can now consider \(\text{Coker} A_{d-1}\) as a \(k[s, t]\) module generated by the natural images of \(x^\alpha y^\beta e_j\) for all non-negative integers \(\alpha, \beta\) and all \(1 \leq j \leq d-1\). The \(k[s, t]\)-module of relations among
these generators is generated by $k[x, y]$-linear combinations of the columns of $A_{d-1}$, and since these columns are bigraded, the $k[s, t]$-module of relations will be bihomogeneous and we can write

$$\text{Coker } A_{d-1} = \bigoplus_{0 \leq D, \ 1 \leq j} (\text{Coker } A_{d-1})_{(D, j)}.$$ 

Consider the $k[s, t]$-module $(\text{Coker } A_{d-1})_{(d, d)}$, the bihomogeneous component of $\text{Coker } A_{d-1}$ of bidegree $(d, d)$. It is generated by the images of $xy^{d-1}e_1, x^2y^{d-2}e_2, \ldots, x^{d-2}y^2e_{d-2}, x^{d-1}y e_{d-1}$ and the relations among these generators are given by $k[s, t]$-linear combinations of $y^{d-2}c_2, xy^{d-3}c_3, \ldots, x^{d-3}y c_{d-1}, x^{d-2}c_d$

where $c_1, \ldots, c_{d+1}$ are the columns of $A_{d-1}$. So we have

$$(\text{Coker } A_{d-1})_{(d, d)} = \text{Coker } B_{d-1}$$

where $B_{d-1}$ is viewed as a $k[s, t]$-homomorphism $k[s, t]^{d-1} \rightarrow k[s, t]^{d-1}$.

Using Lemma 1.1(i) we deduce that for all $d \geq 2$ the direct summand $(\text{Coker } A_{d-1})_{(d, d)}$ of $\text{Coker } A_{d-1}$ has $\tau_{d-1}$-torsion, and so does $\text{Coker } A_{d-1}$ itself.

Lemma 1.1(ii) applied with $S = \mathbb{N}$ now shows that there exist infinitely many irreducible homogeneous polynomials $\{p_i \in k[s, t] : i \geq 1\}$ each one of them contained in some associated prime of the $R_0$-module $\oplus_{d \geq 2} \text{Coker } A_{d-1}$. Clearly, if $i \neq j$ then any prime ideal $P \subset R_0$ which contains both $p_i$ and $p_j$ must contain both $s$ and $t$.

Since the localisation of $(\text{Coker } A_{d-1})_{(d, d)}$ at $s$ does not vanish, there exist $P_i, P_j \subset \text{Ass}_{R_0} \text{Coker } A_{d-1}$ which do not contain $s$ and such that $p_i \subset P_i$, $p_j \subset P_j$, and the previous paragraph shows that $P_j \neq P_j$.

The second statement now follows from the fact that $H^2_{R_1}(R)$ is $R_0$-isomorphic to $\oplus_{d \geq 2} \text{Coker } A_{d-1}$.

\[\square\]

1.3. Corollary. Let $T$ be the localisation of $R$ at the irrelevant maximal ideal $m = \langle s, t, x, y, u, v \rangle$. Then $H^2_{(u,v)T}(T)$ has infinitely many associated primes.

Proof. Since $\tau_i \in m$ for all $i \geq 1$, $H^2_{(u,v)T}(T) \cong (H^2_{(u,v)R}(R))_m$ has $\tau_i$-torsion for all $i \geq 1$.  

\[\square\]

2. A connection with associated primes of Frobenius powers

In this section we apply a technique similar to the one used in section 1 to give a proof of a slightly more general statement of Theorem 12 in [K]. The new proof is simpler, open to generalisations and
it gives a connection between associated primes of Frobenius powers of ideals and of local cohomology modules, at least on a purely formal level.

Let $k$ be any field, let $S = k[x, y, s, t]$, let $F = xy(x - y)(sx - ty) = sx^3y - (t + s)x^2y^2 + txy^3$ and let $R = S/FS$.

2.1. **Theorem.** Let $S$ be an infinite set positive integers and suppose that either $k$ has characteristic zero or that $k$ has characteristic $p$ and that $S$ contains infinitely many powers of $p$. The set

$$\bigcup_{n \in S} \text{Ass}_R \left( \frac{R}{(x^n, y^n)} \right)$$

is infinite.

**Proof.** We introduce a grading in $S$ by setting $\text{deg}(x) = \text{deg}(y) = 1$ and $\text{deg}(s) = \text{deg}(t) = 0$. Since $F$ is homogeneous, $R$ is also graded.

Fix some $n > 0$ and consider the graded $R$-module $T = R/(x^n, y^n)$. For each $d > 4$ consider $T_d$, the degree $d$ homogeneous component of $T$, as a $k[s, t]$-module. If $d < n$, $T_d$ is generated by the images of $y^d, xy^{d-1}, \ldots, x^{d-1}y, x^d$ and the relations among these generators are obtained from $y^{d-4}F, xy^{d-5}F, \ldots, x^{d-5}yF, x^{d-4}F$. Using these generators and relations, in the given order, we write $T_d = \text{Coker} M_d$ where

$$M_d = \begin{pmatrix}
0 & 0 & \ldots & 0 \\
t & -t - s & t & \\
-s & t & -t - s & \\
\vdots & \ddots & \ddots & \ddots \\
-0 & t & -t - s & s \\
0 & 0 & \ldots & 0 
\end{pmatrix}.$$

When $d = n$, $T_d$ is isomorphic to the cokernel of the submatrix of $M_d$ obtained by deleting the first and last rows which correspond to the generators $y^n, x^n$ of $T_n$.

When $d = n + 1$, $T_d$ is isomorphic to the cokernel of the submatrix of $M_d$ obtained by deleting the first two rows and last two rows which correspond to the generators $y^{n+1}, xy^n, x^ny, x^{n+1}$ of $T_{n+1}$, and the resulting submatrix is $B_{n-2}$ defined in Lemma 1.1; the result now follows from that lemma.

$\square$
This technique for finding associated primes of non-finitely generated graded modules and of sequences of graded modules has been applied in [BKS] and [KS] to yield further new and surprising properties of top local cohomology modules.

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References


