APPLICATION OF A HYBRID WKB-GALERKIN METHOD TO A NONLINEAR PLATE DYNAMIC PROBLEM WITH TIME DEPENDENT DAMPING COEFFICIENT

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Abstract

The objective of the analysis is to obtain a closed form approximate analytical solution for the nonlinear differential equation of a loaded plate considering a time variant damping coefficient. The solution of the problem is obtained by using perturbation and a hybrid WKB-Galerkin method. Results are presented of comparison of the solutions based on different approaches.

Keywords

Nonlinear dynamic problem, perturbation method, a hybrid WKB-Galerkin method, time dependent damping coefficient.

1 Introduction

Many physical phenomena involving oscillation cannot be represented in terms of linear theory. Thus non-linear theory potentially enables a description of phenomena which are otherwise hidden for the problem in the context of linear theory.

A special branch of oscillation theory is devoted to non-linear oscillations with specific properties. Such kinds of motion can be observed in plates and shells with large displacements if the strains and displacement are non-linearly related. This class of problem is an integral part of non-linear deformable solid mechanics.

As a rule, investigation in terms of non-linear theory results in nonlinear differential equations. Analytical solution of these equations, especially nonlinear differential equations with variable coefficients, as in this paper, causes many mathematical problems. Therefore, a wide variety of approximate methods such as perturbation techniques, the method of multiple scales and the averaging method, have been proposed for nonlinear differential equations in mechanics. However, it should be noticed that most nonlinear differential equations which are solved by applying such approximate methods are differential equations with constant coefficients.

In this work a problem is presented of plate oscillation which is described by a more general nonlinear differential equation with variable coefficients. The perturbation technique and a hybrid (Wentzel-Kramer-Brillouin) WKB-Galerkin method are used to solve this problem.

The hybrid WKB-Galerkin method enables especially good results to be obtained for approximate solution of a singular differential equation which contains a parameter multiplying the highest order derivative. According to the specific procedure, presented in [Gristchak, Dmitrieva, 1995; Gristchak, Ganilova, 2006], solution of the linear differential equations is conducted in two stages: initially by obtaining the WKB-solution of the problem and then by application of the Bubnov-Galerkin orthogonality procedure, taking asymptotic coefficients into consideration. However, typically, the algebra for the WKB method becomes more tedious as higher order terms are computed, and frequently the work required rises so fast from term to term that even with computational assistance very few terms can be computed. Thus for cases where higher order terms may have significant effect, it is important to get as much use of the information contained in the lower order terms as possible. The hybrid WKB-Galerkin method seems to extend greatly the power and usefulness of the WKB method [Steele, 1971; Steele, 1989] without significant computational effort.

Hybrid methods have proved to be useful in a wide variety of applications such as structural mechanics problems, applications to slender-bodies and thermal problems [Geer, Andersen, 1989; Geer, Andersen, 1990; Geer, Andersen, 1991; Gristchak, Dmitrieva, 1995; Gristchak, Ganilova, 2006]. Significantly, according to results obtained in different branches of mechanics the hybrid WKB-Galerkin method shows a higher accuracy of solution compared to the perturbation and WKB methods.

2 Fundamentals and Solution of the Problem

According to [Cartmell, 1990] there are two categories of non-linearities which can appear in real systems. The first one is based on careful mathematical modeling, when linearization is not applied. In this way large deflection problems are considered. The second category consists of non-linearities which occur because of the description of specific physical phenomenon, such as nonlinear damping generated by material composition.

According to the publication by [Volmir, 1972] in the class of plate oscillation problems of interest here the first category of non-linearities is of current importance. In this case the natural oscillation of a simply supported plate $a \times b$ is described by the following equation:

$$\omega^2 f''(\tau) + \omega_0^2 f(\tau) + \chi f^3(\tau) = 0 \qquad (1)$$

It should be noticed that this equation has been considered and solved in a number of works with a use of perturbation method.

However, the study conducted in [Nayfeh, 1981] reveals the limitations of the perturbation method. These limitations are based on the presence of secular terms and absence of clear relations between the frequency and the power of the nonlinearity. As a result for such kind of nonlinear differential equations, typically of Duffing equation form, the Lindstedt-Poincare technique, the method of multiple scales and the averaging method were all applied.

In this paper we intend to extend the nonlinear equation which describes the plate oscillation under external dynamic loading *q(t)*. Thus we consider a simply supported plated subjected to a dynamic loading *q(t)* and we also take into account a damping coefficient expressed as a time variant function $\varepsilon(t)$. As a result of application of the Bubnov-Galerkin method according to [Volmir, 1972], it is possible to obtain the following equation [Nayfeh, 1981; Volmir, 1972; Gristchak, Kabak, 1996]:

$$\frac{d^2 f^*}{dt^{*2}} + 2\varepsilon(t)\frac{df^*}{dt^*} + K_1 f^* + K_3 f^{*3} = q_0^* \cos\omega^* t^* \qquad (2)$$

According to the procedure described in [Nayfeh, 1981] the problem can be nondimensionalized by introducing the following expressions:

$$t = \frac{t^*}{T^*}, \quad \hat{f} = \frac{f^*}{F^*}. \qquad (3)$$

We also define $K_1 T^{*2} = C$, where C is an integer constant. Thus $T^* = \sqrt{\frac{C}{K_1}} = \frac{\sqrt{C}}{\omega_0^*}$ where ω_0^* is the natural frequency of the appropriate linear problem.

Finally, we make the following definitions

$$\bar{q}_0 = \frac{q_0^* K T^{*2}}{F^*}, \quad \omega = \omega^* T^* = \sqrt{C}\frac{\omega^*}{\omega_0^*}, \quad \lambda^2 = K,$$

$$\beta = CK, \quad \bar{\varepsilon}(t) = \mu K \varepsilon(t),$$

$$\alpha = K K_3 T^{*2} F^{*2} = K^2 \omega_0^{*2} \frac{C}{\omega_0^{*2}} F^{*2} = C K^2 F^{*2}, \qquad (4)$$

where K is a small term determined in [Volmir, 1972] and μ is a positive constant given by [Nayfeh, 1981].

Therefore the equation of interest (2) can be written as follows

$$\lambda^2 \hat{f}''(t) + 2\bar{\varepsilon}(t)\hat{f}'(t) + \beta \hat{f}(t) + \alpha \hat{f}^3(t) = \bar{q}_0 \cos\omega t \qquad (5)$$

It should be noticed that investigation of a problem similar to (5) was conducted by applying the perturbation method in [Nayfeh, 1981] where a nonlinear differential equation with constant coefficients was obtained.

For the solution of the nonlinear equation in (5) the perturbation method is applied, introducing

$$\hat{f}(t,\alpha) = \hat{f}_0(t) + \alpha \hat{f}_1(t) + \ldots \qquad (6)$$

where α is a small parameter.

Substitution of (6) into (5) gives the following equation

$$\begin{aligned} &\lambda^2 \hat{f}_0''(t) + 2\bar{\varepsilon}(t)\hat{f}_0'(t) + \beta\hat{f}_0(t) + \alpha\Big[\lambda^2 \hat{f}_1''(t) + \\ &+ 2\bar{\varepsilon}(t)\hat{f}_1'(t) + \beta\hat{f}_1(t) + \hat{f}_0^3(t)\Big] + \ldots = \bar{q}_0 \cos\omega t \end{aligned} \qquad (7)$$

By equating the terms of like powers it is possible to obtain the following system of nonhomogenous linear differential equations

$$\begin{cases} \lambda^2 \hat{f}_0''(t) + 2\bar{\varepsilon}(t)\hat{f}_0'(t) + \beta\hat{f}_0(t) = \bar{q}_0 \cos\omega t \\ \lambda^2 \hat{f}_1''(t) + 2\bar{\varepsilon}(t)\hat{f}_1'(t) + \beta\hat{f}_1(t) = -\hat{f}_0^3(t) \end{cases} \qquad (8)$$

Solution of these two differential equations can be obtained by applying a hybrid WKB-Galerkin method to the following general differential equation

$$\lambda^2 f''(t) + 2\bar{\varepsilon}(t) f'(t) + \beta f(t) = \tilde{q}(t) \qquad (9)$$

The solution of this equation can be expressed as

$$f(t) = f_c(t) + f_p(t) \qquad (10)$$

where $f_c(t)$ is a complementary function and $f_p(t)$ is a particular solution.

To find the complementary function we solve the following homogeneous differential equation with variable coefficients

$$\lambda^2 f''(t) + 2\bar{\varepsilon}(t) f'(t) + \beta f(t) = 0 \tag{11}$$

where λ^2 is a parameter multiplying the highest order derivative.

According to the hybrid method described in [Gristchak, Dmitrieva, 1995; Gristchak, Ganilova, 2006], taking into consideration only the first two terms, the WKB-solution has the following form

$$f(t,\lambda) = \exp\left[\int_a^t \left(\frac{1}{\lambda} f_0(t) + f_1(t)\right) dt\right] \tag{12}$$

Substituting (12) into equation (11) we obtain

$$\begin{aligned}&\lambda^2\left[\frac{1}{\lambda} f_0' + f_1' + \frac{1}{\lambda^2} f_0^2 + f_1^2 + 2\frac{1}{\lambda} f_0 f_1\right] + \\ &+ 2\bar{\varepsilon}(t)\left[\frac{1}{\lambda} f_0 + f_1\right] + \beta = 0\end{aligned} \tag{13}$$

Equating the coefficients of like powers, we get the system of equations

$$\begin{cases} f_0^2 + 2\bar{\varepsilon}(t) f_1 + \beta = 0 \\ f_0' + 2 f_0 f_1 = 0 \end{cases} \quad \text{or} \quad \begin{cases} f_1 = -\dfrac{1}{2}\dfrac{d}{dt}\ln f_0 \\ \bar{\varepsilon}(t) f_0' - f_0^3 - \beta f_0 = 0 \end{cases} \tag{14}$$

The alternative system (14) can be solved using the standard substitution

$$f_0 = U(t)V(t) \tag{15}$$

This leads to the requirement to solve the following equation

$$\bar{\varepsilon}(t)\left(U'V + UV'\right) - U^3V^3 - \beta UV = 0 \tag{16}$$

from which we obtain

$$V = e^{\beta \int \frac{dt}{\bar{\varepsilon}(t)}} \quad \text{and} \quad U = \pm i\left[2\int \frac{V^2}{\bar{\varepsilon}(t)} dt\right]^{-1/2} \tag{17}$$

Therefore, by considering (17), the solution of the system of equations (14) can be expressed as

$$\begin{cases} f_1 = -\dfrac{1}{2}\dfrac{d}{dt}\ln f_0 \\ f_0 = \pm i\left[2\displaystyle\int \frac{e^{2\beta\int\frac{dt}{\bar{\varepsilon}(t)}}}{\bar{\varepsilon}(t)} dt\right]^{-1/2} e^{\beta\int\frac{dt}{\bar{\varepsilon}(t)}} \end{cases} \tag{18}$$

Thus, the solution based on the application of the WKB method can be defined by (12), taking into consideration (18).

In the second step of the hybrid WKB-Galerkin method, the Bubnov-Galerkin technique was applied. Using only the first term of the WKB-solution (f_0), we consider the solution in the form

$$\tilde{f}(t,\lambda) = \exp\left[\int_a^t \left[\delta_{01}(\lambda) + i\delta_{02}(\lambda)\right] f_0(t) dt\right] \tag{19}$$

According to the hybrid approach, expression (19) can then be substituted into equation (11).

$$\begin{aligned}&\lambda^2\left[\left(\delta_{01} + i\delta_{02}\right) f_0' + \left(\delta_{01}^2 + 2i\delta_{01}\delta_{02} - \delta_{02}^2\right) f_0^2\right] + \\ &+ 2\bar{\varepsilon}(t)\left[\delta_{01} + i\delta_{02}\right] f_0 + \beta = 0\end{aligned} \tag{20}$$

Following the necessary condition of orthogonality of R and the $N+1$ coordinate functions in the interval $[a,b]$,i.e.

$$\int_a^b R\left(\delta_0,\ldots,\delta_N, f_0,\ldots,f_N, f_0',\ldots,f_N^{(n-1)}, t, \lambda\right) f_i(t) dt = 0,$$

we obtain

$$\begin{aligned}&\lambda^2\left[\left(-\delta_{01} - i\delta_{02}\right)\bar{f}_0'\bar{f}_0 + \left(-\delta_{01}^2 i + 2\delta_{01}\delta_{02} + i\delta_{02}^2\right)\left(\pm \bar{f}_0^3\right)\right] + \\ &+ 2\bar{\varepsilon}(t)\left[-\delta_{01} - i\delta_{02}\right]\bar{f}_0^2 + i\beta\left(\pm \bar{f}_0\right) = 0\end{aligned} \tag{21}$$

where $f_0 = \pm i\bar{f}_0$

Equating the coefficients of the *Real* and *Imaginary* terms, we get the following system of equations

$$\begin{cases} A\delta_{02} - B\delta_{01}^2 + B\delta_{02}^2 + W = 0 \\ A\delta_{01} + 2B\delta_{01}\delta_{02} = 0 \end{cases} \tag{22}$$

where

$$\begin{aligned}&A = \int_a^b \left[-\lambda^2 \bar{f}_0'\bar{f}_0 - 2\bar{\varepsilon}(t)\bar{f}_0^2\right] dt; \quad B = \pm\int_a^b \lambda^2 \bar{f}_0^3 dt; \\ &W = \pm\int_a^b \beta\bar{f}_0 dt\end{aligned} \tag{23}$$

Solving the system of equations (22) we obtain

$$\begin{cases} \delta_{01} = \dfrac{\sqrt{4BW - A^2}}{2B} \\ \delta_{02} = \mp\dfrac{A}{2B} \end{cases} \tag{24}$$

Therefore, the solution based on the hybrid WKB-Galerkin method can be expressed as (19) taking into consideration (18) and (24).

It should be noticed that the hybrid solution obtained in the form of (19) is a complementary function of equation (9). To get the particular solution, applying the variation of parameters method, we suppose that

$$\varepsilon(t) = Ct \tag{25}$$

where C is an arbitrary constant.

Taking into consideration assumptions (18), (24) and (25), the hybrid solution (19) can be rewritten as

$$\tilde{f} = e^{-\delta_{02}\sqrt{\beta t}}\left(c_1 \sin \delta_{01}\sqrt{\beta t} + c_2 \cos \delta_{01}\sqrt{\beta t}\right) \tag{26}$$

where c_1, c_2 are arbitrary constants.

Thus the particular solution of equation (9) can be obtained in the form

$$f_p = e^{-\delta_{02}\sqrt{\beta t}}\left(\bar{c}_1 \sin \delta_{01}\sqrt{\beta t} + \bar{c}_2 \cos \delta_{01}\sqrt{\beta t}\right) \tag{27}$$

where $\bar{c}_1 = \int \dfrac{\tilde{q}(t) e^{\delta_{02}\sqrt{\beta t}} \cos \delta_{01}\sqrt{\beta t}}{\lambda^2 \delta_{01}\sqrt{\beta}} dt$,

$\bar{c}_2 = -\int \dfrac{\tilde{q}(t) e^{\delta_{02}\sqrt{\beta t}} \sin \delta_{01}\sqrt{\beta t}}{\lambda^2 \delta_{01}\sqrt{\beta}} dt$.

According to (10), the general solution of equation (9) can be expressed as

$$\begin{aligned} f(t) = e^{-\delta_{02}\sqrt{\beta t}}\Big((\bar{c}_1 + c_1)\sin \delta_{01}\sqrt{\beta t} + \\ +(\bar{c}_2 + c_2)\cos \delta_{01}\sqrt{\beta t}\Big) \end{aligned} \tag{28}$$

3 Numerical Results

To validate the solution obtained, it is important to present graphical results for the problem for predetermined parameters. It is assumed that a plate ($a = b = 2m$) is simply supported and that the parameters described in (4) are given as the following data values

$$\begin{aligned} &q_0^* = 1, \quad T^* = 0.1, \quad F^* = 0.1, \quad \omega = \sqrt{C}\,\frac{\omega^*}{\omega_0^*} = 4, \\ &\lambda^2 = K = 0.91, \quad C = 1.5, \quad \mu = 1, \\ &\alpha = 0.0124215, \quad h = 0.2m \end{aligned} \tag{29}$$

where h is thickness of the plate. It should be noted that the damping coefficient is assumed to be $\varepsilon(t) = 0.1t$ and also $\varepsilon(t) = 0.2t$.

To illustrate graphically the behaviour of closed form solution (28) for the data used in (29), the MAPLE software was applied.

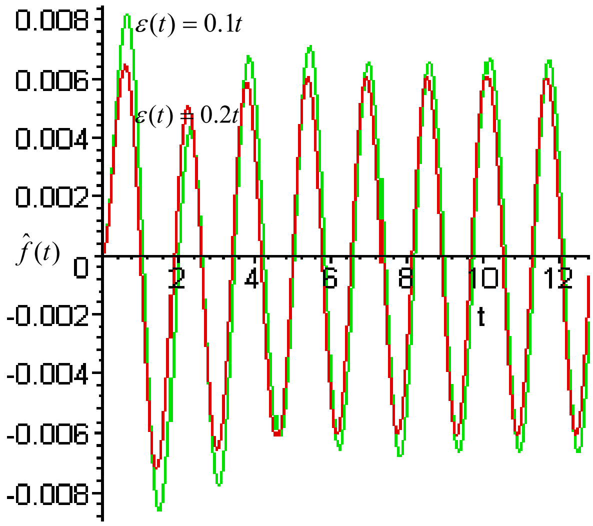

Figure 1. Behaviour of the $\hat{f}(t)$ solution for different values of the time dependent damping coefficient.

To study the correlation of numerical solution of the described problem (5) and the solution obtained above, the Runge-Kutta method was applied. Results of the solution presented in (28) and the numerical solution are presented in the following Table.

Table 1

t	*Runge-Kutta method*	*Solution obtained* $\hat{f}(t)$
0.01	$0.499925385359 \cdot 10^{-5}$	$0.4982645218 \cdot 10^{-5}$
0.02	0.0000199880691487	0.00001985571983
0.03	0.0000449396481076	0.00004449469328
0.04	0.0000798092173662	0.00007875901363
0.05	0.0001245339054669	0.0001224922293
0.1	0.0004925795616544	0.0004766441453
0.2	0.0018832699432609	0.001763702803
0.3	0.0039251810216846	0.003554273261
0.4	0.0062519955355181	0.005460680060
0.5	0.0084380954296054	0.007077298897
1.5	-0.0077429716006232	-0.008593388163
5.5	0.0077334293708881	0.006828513454
10	0.0055302545933302	0.005622470552

To corroborate the solution obtained above, the correlation of the solution of the same problem presented in [Nayfeh, 1981], but for constant damping coefficient, and solution (28) for the time variant damping coefficient is presented. It should be emphasized that the solution for the problem in [Nayfeh, 1981] (*u(t)*) is obtained by applying the perturbation method, as in this paper, but it is not valid for large values of time because of secular terms in the closed form solution. In contrast to the solution from [Nayfeh, 1981], the closed form solution obtained in this work does not consist of secular terms, and therefore can be applied for large values of time. However, for correlation of these two solutions in our numerical example we use only small values of the time parameter, in order to keep the solution presented in [Nayfeh, 1981] valid, and for a constant damping coefficient. Thus, according to [Nayfeh, 1981], we suppose $C = 1$ and $\varepsilon = \varepsilon(t) = 0.0091$, $\alpha = 0.008281$. All other parameters are as stated in (29).

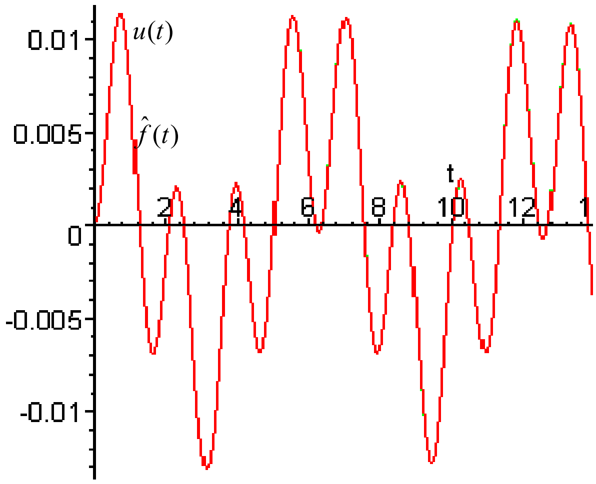

Figure 2. Correlation of solutions presented in [Nayfeh, 1981] *u(t)* and obtained in this work $\hat{f}(t)$.

It is obvious that the functions presented in Figure 2 almost completely coincide. To study the difference in the results the following Table is constructed.

Table 2

t	*Solution u(t), [Nayfeh, 1981]*	*Solution obtained* $\hat{f}(t)$
0.05	0.0001245216182	0.0001245200582
0.1	0.0004926601094	0.0004926539960
0.15	0.001088580582	0.001088567247
0.2	0.001886750012	0.001886727300
0.25	0.002852960166	0.002852926642
0.3	0.003945697896	0.003945652973
0.4	0.006318383566	0.006318317967
0.5	0.008594850248	0.008594773688
0.6	0.01037139786	0.01037132765
0.7	0.01131477692	0.01131473610
0.8	0.01121534388	0.01121535801
0.9	0.01002077994	0.01002087309
1	0.007845043508	0.007845234315

4 Concluding Remarks

Analyzing the results obtained for different values of the dynamic damping coefficient, presented in Figure 1, it is obvious that the damping of the oscillation occurs with an increasing damping coefficient.

The solution obtained in this paper is compared with the numerical solution of the same problem, i.e. considering the time variant damping coefficient, and with the solution presented in [Nayfeh, 1981] for a time invariant damping coefficient, i.e. for $\varepsilon = 0.0091$. According to Table 1, Table 2 and Figure 2 it is obvious that the relative error is almost negligible for all cases. It should be noticed that the solution for the problem in [Nayfeh, 1981] obtained by applying a perturbation method, as in this paper, is not valid for large values of time because of secular terms in the solution function. In contrast to the [Nayfeh, 1981] solution, the closed form solution obtained in this work does not consist of secular terms and can be applied just as efficiently for large values of time.

The comparisons conducted confirm the effectiveness of the hybrid WKB-Galerkin method in this field, and the solution obtained for a more general extended problem of plate oscillation.

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