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Time Bounds for Iterative Auctions: A Unified Approach by Discrete Convex Analysis

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Abstract

We investigate an auction model where there are many different goods, each good has multiple units and bidders have gross substitutes valuations over the goods. We analyze the number of iterations in iterative auction algorithms for the model based on the theory of discrete convex analysis. By making use of $\mathrm{L}^{\natural}$-convexity of the Lyapunov function we derive exact bounds on the number of iterations in terms of the ℓ_∞-distance between the initial price vector and the found equilibrium. Our results extend and unify the price adjustment algorithms for the multi-unit auction model and for the unit-demand auction model, offering computational complexity results for these algorithms, and reinforcing the connection between auction theory and discrete convex analysis.

Keywords: discrete optimization, submodular function, discrete convex function, Walrasian equilibrium, iterative auction

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1. Introduction

In recent years, there has been a growing use of iterative auctions for selling items such as spectrum licenses in telecommunication, electrical power, landing slots at airports, etc. (see [8, 9] for surveys). In such auctions, given a set of discrete (or indivisible) items the auctioneer aims at finding an efficient allocation of items to bidders as well as market clearing prices of the items.

In this paper, we consider a model where there are multiple indivisible goods for sale and each good may have several units; this is more general than the single-unit model treated extensively in the literature (see, e.g., [13, 14, 15, 17, 31]). We are particularly interested in precise time bounds of iterative auctions. Theoretical bounds on the number of iterations are interesting in their own right but also important in practice, providing market participants with an a priori guarantee for the time required to execute a planned auction. While computer simulations are often used to evaluate the practical performance of iterative auctions (see, e.g., [6, 27]), there are only a few scattered results on theoretical analysis of the time complexity so far (see, e.g., [3, 28]).

The objective of this paper is to provide a unified method of analysis for iterative auctions based on the theory of discrete convex analysis. Our contribution consists of the following two aspects.

In the multi-unit auction model with (strong) gross substitutes valuations, Ausubel [4] proposed several iterative auctions, all of which are based on minimization of a function called the Lyapunov function. Our first contribution is to reveal a nice combinatorial property of the Lyapunov function—discrete convexity ($\mathrm{L}^\natural$-convexity), and analyze the number of iterations required in iterative auctions by utilizing the theory of discrete convex analysis. $\mathrm{L}^\natural$-convexity of the Lyapunov function is shown by using the discrete conjugacy theorem in discrete convex analysis and the equivalence between the (strong) gross substitutes condition and some discrete concavity ($\mathrm{M}^\natural$-concavity) due to Fujishige and Yang [12] and Murota and Tamura [26] (see Section 4.1). We give the exact bounds for the ascending and descending auctions in [4] and their variants in terms of the ℓ_∞-distance between the initial price vector and the equilibrium price vector (see Theorems 4.7, 4.8, 4.10, and 4.11). This implies, in particular, that the trajectory of the price vector generated by the ascending or descending auction is the "shortest" path between the initial vector and the equilibrium price vector. This result also exhibits an appealing feature of the ascending and descending auctions. Another iterative auction named the two-phase auction, consisting of a single ascending phase and a single descending phase, is also considered in this paper (see Theorems 4.12 and 4.13 and Remark 4.14).

Our second contribution is concerned with the *unit-demand auction model* in the sense that each bidder is interested in getting at most one item. Iterative auctions for this model are discussed extensively in the literature (see, e.g., [2, 3, 10, 18, 19, 28]). Specifically, Vickrey–English auction by Demange et al. [10], Vickrey–Dutch auction by Mishra and Parkes [18], and Vickrey–English–Dutch auction by Andersson and Erlanson [3] are such iterative auctions. These three algorithms are proposed independently of the iterative auction algorithms for the multi-unit model. We offer a unified treatment of these iterative auction algorithms by revealing their relationship to the general iterative auction algorithms for the general model. In particular, we show that the sequence of price vectors generated by Vickrey–English auction (resp., Vickrey–Dutch auction) coincides with that generated by an ascending auction algorithm (resp., a descending auction algorithm) when applied to unit-demand auctions. This observation, combined with our first contribution described above, yields immediately the known bounds for the number of iterations in Vickrey–English auction and Vickrey–Dutch auction. A new bound for Vickrey–English–Dutch auction is obtained from our result for the two-phase auction algorithm above.

The organization of this paper is as follows. In Section 2, we explain auction models and fundamental concepts used in this paper. In Section 3 we review the concept of discrete convexity and some fundamental results in discrete convex analysis. In Section 4, we analyze the number of iterations required in iterative auctions in the multi-unit model with gross substitutes valuations, while iterative auction algorithms for the unit-demand auctions are discussed in Section 5.

This paper is the full version of our extended abstract [25] that appeared in the conference proceedings of ISAAC 2013, with a substantial extension described in Section 5.

2. Fundamental Concepts in Auctions

We explain auction models and fundamental concepts used in this paper.

2.1. Auction Models and Walrasian Equilibrium.

In the auction market, there are n types of items or goods, denoted by $N = \{1, 2, \dots, n\}$, and m bidders, denoted by $M = \{1, 2, \dots, m\}$, where $m \geq 2$. We have $u(i)$ units available for each item $i \in N$, where $u(i)$ is a positive integer. We denote the integer interval as $[\mathbf{0}, u]_{\mathbb{Z}} = \{x \in \mathbb{Z}^n \mid \mathbf{0} \leq x \leq u\}$,

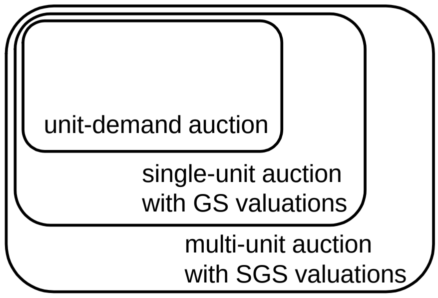

Figure 1: Relationship among three auction models.

where $u = (u(1), u(2), \dots, u(n))^\top$. Each vector $x \in [\mathbf{0}, u]_\mathbb{Z}$ is called a *bundle*; a bundle $x = (x(1), x(2), \dots, x(n))^\top$ corresponds to a (multi-)set of items, where $x(i)$ represents the multiplicity of item $i \in N$. Each bidder $j \in M$ has his valuation function $f_j : [\mathbf{0}, u]_\mathbb{Z} \to \mathbb{R}$; the number $f_j(x)$ represents the value of the bundle x worth to bidder j. The case with $u(i) = 1$ for all $i \in N$ is referred to as *single-unit auction* in this paper, while the case with general u as *multi-unit auction*. Note that $[\mathbf{0}, \mathbf{1}]_\mathbb{Z} = \{0, 1\}^n$, where $\mathbf{1} = (1, 1, \dots, 1)^\top$. A further special case where each bidder is interested in getting at most one item is called *unit-demand auction*; see Section 5 for more detailed description of this auction model. The relationship among the three auction models is summarized in Figure 1.

In an auction, we want to find an efficient allocation and market clearing prices. An *allocation* of items is defined as a set of bundles $x_1, x_2, \dots, x_m \in [\mathbf{0}, u]_\mathbb{Z}$ satisfying $\sum_{j=1}^m x_j = u$. Given a price vector $p \in \mathbb{R}^n_+$, each bidder $j \in M$ wants to have a bundle x which maximizes the value $f_j(x) - p^\top x$. For $j \in M$ and $p \in \mathbb{R}^n_+$, define

$$V_j(p) = \max\{f_j(x) - p^\top x \mid x \in [\mathbf{0}, u]_\mathbb{Z}\}, \tag{2.1}$$

$$D_j(p) = \arg\max\{f_j(x) - p^\top x \mid x \in [\mathbf{0}, u]_\mathbb{Z}\}. \tag{2.2}$$

We call the function $V_j : \mathbb{R}^n_+ \to \mathbb{R}$ and the set $D_j(p) \subseteq [\mathbf{0}, u]_\mathbb{Z}$ the *indirect utility function* and the *demand set*, respectively. The auctioneer wants to find a pair of a price vector p^* and an allocation $x_1^*, x_2^*, \dots, x_m^*$ such that $x_j^* \in D_j(p^*)$ for $j \in M$. This pair is called a *(Walrasian) equilibrium*; p^* is a *(Walrasian) equilibrium price vector* (see, e.g., [8, 9]). Thus, in an equilibrium every bidder gets an optimal bundle for himself and all goods are sold; i.e., all market participants are in harmony.

Although the Walrasian equilibrium possesses a variety of desirable properties, it does not always exist. Hence, some assumption for bidders' valuation functions is required to guarantee the existence of a Walrasian equilibrium.

Remark 2.1. Two different definitions of Walrasian equilibrium can be found in the literature. The definition in this paper follows the one in [4]. An alternative definition, in which unsold items may exist, is as follows (cf. [13, Section 3]): a pair of a price vector p^* and a set of bundles $\tilde{x}_1, \ldots, \tilde{x}_m^*$ is called a *(Walrasian) equilibrium* if it satisfies the following condition:

$$\left.\begin{array}{l} \sum_{j=1}^m \tilde{x}_j \leq u, \\ \tilde{x}_j \in D_j(p^*) \text{ for all } j \in M, \\ p^*(i) = 0 \text{ for all } i \in N \text{ with } \sum_{j=1}^m \tilde{x}_j(i) < u(i). \end{array}\right\} \qquad (2.3)$$

The two definitions of equilibrium are essentially equivalent in the sense that the set of equilibrium price vectors remains the same under the two definitions, provided that valuation functions f_j $(j \in M)$ are nondecreasing (see, e.g., [13, Section 3]). Indeed, it is easy to see that if a pair of a price vector p^* and an allocation $x_1^*, x_2^*, \ldots, x_m^*$ is an equilibrium in our sense, then the pair also satisfies the condition (2.3). Conversely, if a pair of a price vector p^* and a set of bundles $\tilde{x}_1, \ldots, \tilde{x}_m^*$ satisfies the condition (2.3), the allocation $x_1^*, x_2^*, \ldots, x_m^*$ given by

$$x_j^* = \tilde{x}_j \ (j = 1, 2, \ldots, m-1), \quad x_m^* = u - \sum_{j=1}^{m-1} \tilde{x}_j$$

satisfies $x_j^* \in D_j(p^*)$ for $j = 1, 2, \ldots, m$; note that $x_m^* \in D_m(p^*)$ follows from the condition (2.3) and the fact that valuation function f_m is nondecreasing. Hence, the pair of the price vector p^* and the allocation $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_m$ is an equilibrium in our sense. $\square$

2.2. Gross Substitutes Condition and Discrete Concavity.

We say that function f_j satisfies the *gross substitutes (GS)* condition if it satisfies the following:

$$\textbf{(GS)} \quad \begin{array}{l} \forall p, q \in \mathbb{R}_+^n \text{ with } p \leq q,\ \forall x \in D_j(p),\ \exists y \in D_j(q) : \\ x(i) \leq y(i) \ (\forall i \in N \text{ with } p(i) = q(i)). \end{array}$$

This condition means that when prices of some items increase, the only items that may drop from the optimal bundle are those with increased prices.

The GS condition is originally introduced by Kelso and Crawford [15] for valuation functions defined on 0-1 vectors in the setting of a fairly general two-sided job matching model. Since then, this condition has been widely used in various models such as matching, housing, and labor markets (see, e.g., [4, 5, 7, 8, 9, 13, 14, 16]).

Various characterizations of GS condition are given in the literature of discrete convex analysis and auction theory [5, 12, 13, 14]. Among others, Fujishige and Yang [12] revealed the relationship between GS condition and discrete concavity called $\mathrm{M}^\natural$-concavity (see Section 3.1 for the definition). The concept of $\mathrm{M}^\natural$-concave function is introduced by Murota and Shioura [23], independently of GS condition, as a class of discrete concave functions. It is an extension of the concept of M-concave function introduced by Murota [20]. The concepts of $\mathrm{M}^\natural$-concavity/M-concavity play primary roles in the theory of discrete convex analysis [21].

It is shown by Fujishige and Yang [12] that GS condition and $\mathrm{M}^\natural$-concavity are equivalent in the case of single-unit auction.

Theorem 2.2 (Fujishige and Yang [12])**.** *A valuation function $f : \{0, 1\}^n \to \mathbb{R}$ defined on* 0-1 *vectors satisfies the GS condition if and only if it is an $M^\natural$-concave function.*

This result initiated a strong interaction between discrete convex analysis and auction theory; the results obtained in discrete convex analysis are used in auction theory ([7, 16], etc.), while auction theory provides discrete convex analysis with interesting applications (see, e.g., [26]).

It is known that in single-unit auctions, a Walrasian equilibrium does exist if bidder's valuation functions satisfy the GS condition. The GS condition, however, is not sufficient for the existence of an equilibrium in multi-unit auctions. In the last decade, several papers independently tried to identify conditions for valuation functions to guarantee the existence of an equilibrium in a multi-unit auction. Murota and Tamura [26] proposed a stronger version of GS condition by using the relationship with $\mathrm{M}^\natural$-concavity, and proved the existence of an equilibrium in a more general setting (see also [21, Chapter 11]).

In this paper, we use the *strong gross substitutes (SGS)* condition given by Milgrom and Strulovici [17] (see also [29, Section 4]). We say that a valuation function satisfies the SGS condition if the function satisfies the GS condition when each unit of items is regarded as being distinct. More precisely, for a valuation function $f : [\mathbf{0}, u]_{\mathbb{Z}} \to \mathbb{R}$, we associate a function

$\tilde{f} : \{0,1\}^{\tilde{N}} \to \mathbb{R}$ by considering

$$\tilde{N} = \{(i, \beta) \mid i \in N,\ 1 \le \beta \le u(i)\}$$

and defining $\tilde{f}(\tilde{x})$ for $\tilde{x} \in \{0,1\}^{\tilde{N}}$ by

$$\tilde{f}(\tilde{x}) = f(x), \qquad \text{where } x(i) = \sum_{\beta=1}^{u(i)} \tilde{x}(i, \beta) \qquad (i \in N). \tag{2.4}$$

Then, by definition, f satisfies the SGS condition if and only if $\tilde{f}$ satisfies the GS condition. The SGS condition turns out to be equivalent to M$^\natural$-concavity (see Theorem 4.1 below) and also to the condition given by Murota and Tamura [26].

Throughout this paper we assume the following conditions for all bidders' valuation functions f_j ($j = 1, 2, \ldots, m$) defined on $[\mathbf{0}, u]_{\mathbb{Z}}$:

(A0) f_j is monotone nondecreasing,
(A1) f_j satisfies the SGS condition,
(A2) f_j takes integer values.

The assumption (A2) can be removed if we only need an ε-approximate equilibrium price vector, which is defined, for $\varepsilon > 0$, as a vector p such that $\|p - p^*\|_\infty < \varepsilon$ for some equilibrium price vector p^*. For such a problem, all results in this paper can be adapted easily with slight modifications.

2.3. Iterative Auctions.

An auction algorithm called the *iterative auction* (or Walrasian tâtonnement process, price adjustment process, dynamic auction, etc.) is studied extensively in the auction literature [8, 9]. An iterative auction finds an equilibrium price vector by iteratively updating a current price vector p using information on demand sets $D_j(p)$.

The most natural and popular iterative auction is *ascending auction*, in which the current price vector is increased monotonically. Ascending auction is a natural generalization of the classical English auction for a single item; in addition, it is natural from the economic point of view, and easy to understand and implement. For single-unit auctions with GS valuation functions, an ascending auction of Gul and Stacchetti [14] can find an equilibrium price vector.

Ausubel [4] featured the *Lyapunov function*, which is defined by

$$L(p) = \sum_{j=1}^{m} V_j(p) + u^\top p \qquad (p \in \mathbb{R}^n), \tag{2.5}$$

where the vector $u \in \mathbb{Z}^n_+$ represents the numbers of available units for items in N. Use of the Lyapunov function is motivated by the fact that the set of excess supply vectors at a price vector p (i.e., the set $\{u - \sum_{j=1}^{m} x_j \mid x_j \in D_j(p)\}$) coincides with the set of subgradients of the Lyapunov function at p. The following important properties of the Lyapunov function are known (see [4, 30]).

Theorem 2.3. *Suppose that all bidders' valuation functions f_j $(j = 1, 2, \ldots, m)$ defined on $[\mathbf{0}, u]_{\mathbb{Z}}$ satisfy the conditions* (A0) *and* (A1).
(i) *A price vector $p \in \mathbb{R}^n$ is an equilibrium price vector if and only if it is a minimizer of the Lyapunov function L.*
(ii) *The minimal equilibrium price vector $\underline{p}^*$ and the maximal equilibrium price vector $\overline{p}^*$ are uniquely determined. Moreover, if the valuation functions f_j are integer-valued (i.e., satisfy* (A2)*), then $\underline{p}^*$ and $\overline{p}^*$ are integral, i.e., $\underline{p}^*, \overline{p}^* \in \mathbb{Z}^n$.*

The ascending auction algorithm in Ausubel [4], which is a reformulation of the ascending auction by Gul and Stacchetti [14], finds the minimal integral minimizer $\underline{p}^*$ of the Lyapunov function in a finite number of iterations by updating the price vector in a greedy manner (see Section 4.2 for details). Ausubel [4] also proposed a descending auction algorithm, which finds the maximal integral minimizer $\overline{p}^*$ of the Lyapunov function by iteratively decreasing the price vector from an initial price vector. While the ascending and descending auction algorithms have various nice properties (see, e.g., [8, 9]), they have a disadvantage that the initial price vector must be a lower (or upper) bound of the equilibrium price vector $\underline{p}^*$ (or $\overline{p}^*$). Ausubel [4] proposed a third iterative auction, named "global Walrasian tâtonnement algorithm," which can start with an arbitrary price vector.

3. Preliminaries from Discrete Convex Analysis

We review the concepts of M$^\natural$-concave and L$^\natural$-convex functions and present some useful properties. See [21] for more account of these concepts.

3.1. Definitions and Conjugacy.

A valuation function $f_j : [\mathbf{0}, u]_{\mathbb{Z}} \to \mathbb{R}$ is said to be *M$^\natural$-concave* (read "M-natural-concave") if it satisfies the following:

$$\textbf{(M}^\natural\textbf{-EXC)}\ \forall x, y \in [\mathbf{0}, u]_{\mathbb{Z}},\ \forall i \in \mathrm{supp}^+(x - y),\ \exists k \in \mathrm{supp}^-(x - y) \cup \{0\} : \\ f_j(x) + f_j(y) \le f_j(x - \chi_i + \chi_k) + f_j(y + \chi_i - \chi_k).$$

Here, we denote

$$\mathrm{supp}^+(x) = \{i \in N \mid x(i) > 0\}, \qquad \mathrm{supp}^-(x) = \{i \in N \mid x(i) < 0\}$$

for a vector $x \in \mathbb{R}^n$, $\chi_i \in \{0,1\}^n$ is the characteristic vector of $i \in N$ (i.e., the i-th unit vector), and $\chi_0 = \mathbf{0} = (0,0,\ldots,0)^\top$.

Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a polyhedral convex function, i.e., a convex function such that the epigraph $\{(p,\alpha) \mid p \in \mathbb{R}^n,\ \alpha \in \mathbb{R},\ g(p) \le \alpha\}$ is a polyhedron. We denote

$$\begin{aligned} \mathrm{dom}\, g &= \{p \in \mathbb{R}^n \mid g(p) < +\infty\}, \\ \arg\min g &= \{p \in \mathbb{R}^n \mid g(p) \le g(q)\ (\forall q \in \mathbb{R}^n)\}. \end{aligned}$$

A polyhedral convex function g is said to be *polyhedral* $L^\natural$*-convex* if for every $p, q \in \mathrm{dom}\, g$ and every nonnegative $\lambda \in \mathbb{R}_+$, it holds that

$$g(p) + g(q) \ge g((p + \lambda\mathbf{1}) \wedge q) + g(p \vee (q - \lambda\mathbf{1})), \tag{3.1}$$

where $\mathbf{1} = (1,1,\ldots,1)^\top$, and for $p, q \in \mathbb{R}^n$, $p \wedge q$ and $p \vee q$ denote, respectively, the vectors obtained by component-wise minimum and maximum of p and q. The property (3.1) is called *translation submodularity.* By (3.1) with $\lambda = 0$, an L$^\natural$-convex function g is a submodular function on $\mathbb{R}^n$, i.e.,

$$g(p) + g(q) \ge g(p \wedge q) + g(p \vee q) \qquad (\forall p, q \in \mathrm{dom}\, g).$$

A polyhedral L$^\natural$-convex function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called *domain-integral* [11, p. 317] if $\arg\min\{g(p) - p^\top x \mid p \in \mathrm{dom}\, g\}$ is an integral polyhedron for every $x \in \mathbb{R}^n$ with $\arg\min\{g(p) - p^\top x \mid p \in \mathrm{dom}\, g\} \neq \emptyset$. It is known that domain-integral polyhedral L$^\natural$-convex functions are closed under addition.

Proposition 3.1 ([21])**.** *The sum of (two or more) domain-integral polyhedral* $L^\natural$*-convex functions is also a domain-integral polyhedral* $L^\natural$*-convex function.*

We note that by the definition of domain-integral polyhedral L$^\natural$-convex function, the minimization of a domain-integral polyhedral L$^\natural$-convex function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ can be reduced to the minimization of g on the integer lattice points $\mathbb{Z}^n$. It is easy to see that the restriction of g on the integer lattice points $\mathbb{Z}^n$ satisfies the inequality (3.1) for every $p, q \in \mathbb{Z}^n$ and every $\lambda \in \mathbb{Z}_+$. In general, a function $g : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is called an $L^\natural$*-convex function* if it satisfies the inequality (3.1) for every $p, q \in \mathbb{Z}^n$ and every $\lambda \in \mathbb{Z}_+$.

The following conjugacy relation holds between M$^\natural$-concavity and L$^\natural$-convexity.

Proposition 3.2 ([21])**.** *Let* $f : [\mathbf{0}, u]_{\mathbb{Z}} \to \mathbb{R}$ *be a function.*
(i) f *is an* $M^{\natural}$*-concave function if and only if the function* $g : \mathbb{R}^n \to \mathbb{R}$ *defined by*

$$g(p) = \max\{f(x) - p^{\top} x \mid x \in [\mathbf{0}, u]_{\mathbb{Z}}\} \qquad (p \in \mathbb{R}^n) \tag{3.2}$$

is a polyhedral $L^{\natural}$*-convex function.*
(ii) f *is an integer-valued* $M^{\natural}$*-concave function if and only if* g *is a domain-integral polyhedral* $L^{\natural}$*-convex function such that* $g(p)$ *is an integer for every* $p \in \mathbb{Z}^n$*.*

3.2. Minimization Algorithms.

We consider minimization of an $\mathrm{L}^{\natural}$-convex function $g : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ defined on the integer lattice points. We denote

$$\mathrm{dom}_{\mathbb{Z}} g = \{p \in \mathbb{Z}^n \mid g(p) < +\infty\},$$
$$\arg\min_{\mathbb{Z}} g = \{p \in \mathbb{Z}^n \mid g(p) \leq g(q) \ (\forall q \in \mathbb{Z}^n)\}.$$

To the end of this section we assume that $\arg\min_{\mathbb{Z}} g$ is nonempty and bounded. It is known that under such assumptions, $\arg\min_{\mathbb{Z}} g$ has the uniquely determined minimal and maximal elements, which we denote by $\underline{q}^*$ and $\overline{q}^*$, i.e.,

$$\underline{q}^* = \text{the (uniquely determined) minimal minimizer of } g,$$
$$\overline{q}^* = \text{the (uniquely determined) maximal minimizer of } g.$$

This minimization problem can be solved by certain greedy (or steepest descent) algorithms [22]. We first consider a greedy algorithm such that the vector p is always increased. For $X \subseteq N$, we denote by $\chi_X \in \{0,1\}^n$ the characteristic vector of X, i.e., $\chi_X(i) = 1$ if $i \in X$ and $\chi_X(i) = 0$ if $i \in N \setminus X$.

Algorithm GreedyUp
Step 0: Set $p := p^{\circ}$, where $p^{\circ} \in \mathrm{dom}_{\mathbb{Z}} g$ satisfies $p^{\circ} \leq q$ for some minimizer q of g.
Step 1: Find a minimizer $X \subseteq N$ of $g(p + \chi_X)$.
Step 2: If $X = \emptyset$, then output p and stop.
Step 3: Set $p := p + \chi_X$ and go to Step 1.

A tight bound of the number of iterations of GreedyUp is known.

Proposition 3.3 ([24, Theorem 1.3])**.** *The algorithm* GreedyUp *terminates by outputting a minimizer q^* of g, and the number of updates of p is exactly equal to $\|q^* - p^\circ\|_\infty$.*

Proof. Theorem 1.3 in [24] implies that GreedyUp outputs a minimizer of g exactly in $\|q^* - p^\circ\|_\infty + 1$ iterations. Since the last iteration in GreedyUp is used to check the optimality of p and does not update p itself, the number of updates of p is equal to $\|q^* - p^\circ\|_\infty$. □

We note that the vector q^* found by GreedyUp satisfies

$$\|q^* - p^\circ\|_\infty = \min\{\|p^* - p^\circ\|_\infty \mid p^* \in \arg\min_{\mathbb{Z}} g,\ p^* \geq p^\circ\}.$$

Hence, Proposition 3.3 shows that the trajectory of the vector p generated by GreedyUp is the "shortest" path between the initial vector p° and the found minimizer q^* of g.

To find the minimal minimizer $\underline{q}^*$ of g, a variant of GreedyUp called GreedyUpMinimal is considered, where Step 0 and Step 1 in GreedyUp are replaced with the following:

Step 0: Set $p := p^\circ$, where $p^\circ \in \mathrm{dom}_{\mathbb{Z}}\, g$ satisfies $p^\circ \leq \underline{q}^*$.
Step 1: Find the minimal minimizer $X \subseteq N$ of $g(p + \chi_X)$.

That is, a minimal X is found in Step 1, which is uniquely determined by the L$^\natural$-convexity of g. GreedyUpMinimal outputs the minimal minimizer $\underline{q}^*$ of g, as shown in the following proposition. In addition, a tight bound of the number of iterations can be given. Proofs of Propositions 3.4, 3.5, 3.6, and 3.7 are given at the end of this section.

Proposition 3.4. *The algorithm* GreedyUpMinimal *terminates by outputting the minimal minimizer $\underline{q}^*$ of g, and the number of updates of p is exactly equal to $\|\underline{q}^* - p^\circ\|_\infty$.*

We consider another variant of GreedyUp called GreedyUpMaximal, where Step 0 and Step 1 in GreedyUp are replaced with the following:

Step 0: Set $p := p^\circ$, where $p^\circ \in \mathrm{dom}_{\mathbb{Z}}\, g$ satisfies $p^\circ \leq \overline{q}^*$.
Step 1: Find the maximal minimizer $X \subseteq N$ of $g(p + \chi_X)$.

That is, a maximal X is found in Step 1 instead of a minimal X, where a maximal minimizer $X \subseteq N$ of $g(p + \chi_X)$ is uniquely determined by the L$^\natural$-convexity of g. This modification makes it possible to output the maximal minimizer of g instead of the minimal one.

Proposition 3.5. *The algorithm* GreedyUpMaximal *terminates by outputting the maximal minimizer $\overline{q}^*$ of g, and the number of updates of p is exactly equal to $\|\overline{q}^* - p^\circ\|_\infty$.*

Symmetrically, we can consider algorithms GreedyDownMaximal and GreedyDownMinimal, where the vector p is always decreased. Due to the $\mathrm{L}^\natural$-convexity of g, minimal and maximal minimizers $X \subseteq N$ of $g(p - \chi_X)$ in Step 1 are uniquely determined.

Algorithm GreedyDownMaximal
Step 0: Set $p := p^\circ$, where $p^\circ \in \mathrm{dom}_{\mathbb{Z}}\, g$ satisfies $p^\circ \geq \overline{q}^*$.
Step 1: Find the minimal minimizer $X \subseteq N$ of $g(p - \chi_X)$.
Step 2: If $X = \emptyset$, then output p and stop.
Step 3: Set $p := p - \chi_X$ and go to Step 1.

Proposition 3.6. *The algorithm* GreedyDownMaximal *terminates by outputting the maximal minimizer $\overline{q}^*$ of g, and the number of updates of p is exactly equal to $\|\overline{q}^* - p^\circ\|_\infty$.*

The algorithm GreedyDownMinimal is the one obtained from Greedy-DownMaximal by replacing Step 0 and Step 1 with the following:

Step 0: Set $p := p^\circ$, where $p^\circ \in \mathrm{dom}_{\mathbb{Z}}\, g$ satisfies $p^\circ \geq \underline{q}^*$.
Step 1: Find the maximal minimizer $X \subseteq N$ of $g(p - \chi_X)$.

Proposition 3.7. *The algorithm* GreedyDownMinimal *terminates by outputting the minimal minimizer $\underline{q}^*$ of g, and the number of updates of p is exactly equal to $\|\underline{q}^* - p^\circ\|_\infty$.*

Proof of Propositions 3.4, 3.5, 3.6, and 3.7. We first prove Proposition 3.4. The behavior of GreedyUpMinimal applied to g is the same as that of GreedyUp applied to the $\mathrm{L}^\natural$-convex function $g_\varepsilon(p) = g(p) + \varepsilon \sum_{i=1}^n p(i)$ with a sufficiently small positive ε. Indeed, we have the following equivalences:

$$\begin{aligned}
&X \subseteq N \text{ is a minimizer of } g_\varepsilon(p + \chi_X)\\
&\qquad \iff X \text{ is the minimal minimizer of } g(p + \chi_X),\\
&p \in \mathbb{Z}^n_+ \text{ is a minimizer of } g_\varepsilon\\
&\qquad \iff p \text{ is the minimal minimizer of } g.
\end{aligned}$$

This fact, together with Proposition 3.3, implies the claim of Proposition 3.4.

The proof Proposition 3.5 is quite similar to that for Proposition 3.4, where the function $g^\varepsilon(p) = g(p) - \varepsilon \sum_{i=1}^n p(i)$ is used instead of g_ε.

Finally, Propositions 3.6 and 3.7 follow immediately from Propositions 3.4 and 3.5, respectively, applied to the $\mathrm{L}^\natural$-convex function $\hat{g}(p) = g(-p)$. $\square$

4. Analysis of Iterative Auctions

In this section, we analyze the number of iterations of several iterative auction algorithms for finding an integral equilibrium price vector.

4.1. L$^\natural$-convexity of Lyapunov Function

We prove L$^\natural$-convexity of the indirect utility functions and the Lyapunov function. This observation plays a key role in the analysis of iterative auctions. We first note the equivalence between the SGS condition and M$^\natural$-concavity.

Theorem 4.1. *A function $f : [\mathbf{0}, u]_{\mathbb{Z}} \to \mathbb{Z}$ satisfies the SGS condition if and only if it is M$^\natural$-concave.*

This theorem can be shown as follows. By definition, the SGS condition for a function $f : [\mathbf{0}, u]_{\mathbb{Z}} \to \mathbb{Z}$ is equivalent to the GS condition for $\tilde{f} : \{0,1\}^{\tilde{N}} \to \mathbb{Z}$ given by (2.4). We can also show the following.

Proposition 4.2. *A function $f : [\mathbf{0}, u]_{\mathbb{Z}} \to \mathbb{Z}$ is M$^\natural$-concave if and only if the function $\tilde{f} : \{0,1\}^{\tilde{N}} \to \mathbb{Z}$ defined by* (2.4) *is M$^\natural$-concave.*

The proof is rather straightforward and therefore omitted. By Theorem 2.2, the function $\tilde{f} : \{0,1\}^{\tilde{N}} \to \mathbb{Z}$ satisfies the GS condition if and only if it is an M$^\natural$-concave function. A combination of this fact with Proposition 4.2 yields Theorem 4.1.

We then prove that L$^\natural$-convexity of the indirect utility function is equivalent to the SGS condition of a valuation function.

Theorem 4.3. *Let $V_j : \mathbb{R}^n \to \mathbb{R}$ be the indirect utility function associated with the valuation function f_j given by* (2.1)*.*
(i) *f_j satisfies the condition* (A1) *(i.e., f_j satisfies the SGS condition) if and only if V_j is a polyhedral L$^\natural$-convex function.*
(ii) *f_j satisfies the conditions* (A1) *and* (A2) *if and only if V_j is a domain-integral polyhedral L$^\natural$-convex function such that $V_j(p)$ is an integer for every $p \in \mathbb{Z}^n$.*

Proof. By Theorem 4.1, the SGS condition and M$^\natural$-concavity is equivalent for f_j. This fact, combined with Proposition 3.2, implies the claims of the theorem. □

Corollary 4.4. *Suppose that all bidders' valuation functions f_j $(j = 1, 2, \ldots, m)$ satisfy the conditions* (A1) *and* (A2)*. Then, the Lyapunov function $L : \mathbb{R}^n \to \mathbb{R}$ in* (2.5) *is a domain-integral polyhedral L$^\natural$-convex function such that $L(p)$*

is an integer for every $p \in \mathbb{Z}^n$. In particular, the minimal and maximal minimizers of the Lyapunov function L are integral vectors.

Proof. The claim follows from Theorem 4.3 and Proposition 3.1. □

On the basis of Corollary 4.4, we regard the Lyapunov function L, originally defined on $\mathbb{R}^n$, as a function on $\mathbb{Z}^n$. That is, the Lyapunov function $L : \mathbb{Z}^n \to \mathbb{R}$ is an L$^\natural$-convex function.

We denote by $\underline{p}^*$ (resp., $\overline{p}^*$) the (uniquely determined) minimal (resp., maximal) integral equilibrium price vector. In the following proposition we give an interval in which $\underline{p}^*$ and $\overline{p}^*$ are guaranteed to exist. Define $a \in \mathbb{Z}^n_+$ by

$$a(i) = \max_{j \in M}\{f_j(\chi_i) - f_j(\mathbf{0})\} \qquad (i \in N). \tag{4.1}$$

Proposition 4.5. *Suppose that all bidders' valuation functions f_j ($j = 1, 2, \ldots, m$) defined on $[\mathbf{0}, u]_{\mathbb{Z}}$ satisfy the conditions* (A0) *and* (A1). *Then, every equilibrium price vector $p \in \mathbb{R}^n$ satisfies $\mathbf{0} \le p \le a$.*

The proof is outlined in Section 4.4.1.

4.2. Ascending and Descending Auction Algorithms

We first consider the ascending auction algorithm of Ausubel [4], which can be described as follows:

Algorithm AscendMinimal
Step 0: Set $p := p^\circ$, where $p^\circ \in \mathbb{Z}^n$ satisfies $p^\circ \le \underline{p}^*$ (e.g., $p^\circ = \mathbf{0}$).
Step 1: Find the minimal minimizer $X \subseteq N$ of $L(p + \chi_X)$.
Step 2: If $X = \emptyset$, then output p and stop.
Step 3: Set $p := p + \chi_X$ and go to Step 1.

Note that the algorithm AscendMinimal can be interpreted in auction terms as follows (see [4, Appendix B] for details about the implementation of Steps 2 and 3):

Step 0: The auctioneer sets $p := p^\circ$, where $p^\circ \in \mathbb{Z}^n$ satisfies $p^\circ \le \underline{p}^*$.
Step 1: The auctioneer asks the bidders to report their demand sets $D_j(p)$ ($j \in M$), and finds the minimal minimizer $X \subseteq N$ of $L(p + \chi_X)$.
Step 2: The auctioneer checks if $X = \emptyset$; if $X = \emptyset$ holds, then the auctioneer reports p as the final price vector and stop.
Step 3: The auctioneer sets $p := p + \chi_X$ and returns to Step 1.

Theorem 4.6 ([4])**.** *Starting from an integral vector* p° *with* $p^\circ \leq \underline{p}^*$*, the algorithm* AscendMinimal *outputs the minimal integral equilibrium price vector* $\underline{p}^*$ *in a finite number of iterations.*

The exact bound for the number of iterations in AscendMinimal is given in terms of the ℓ_∞-distance between the initial price vector and the minimal equilibrium price vector $\underline{p}^*$.

Theorem 4.7. *Suppose that the algorithm* AscendMinimal *starts from an integral vector* p° *with* $p^\circ \leq \underline{p}^*$*. Then, the number of updates of the price vector is exactly equal to* $\|\underline{p}^* - p^\circ\|_\infty$*.*

Proof. The Lyapunov function L is an L$^\natural$-convex function by Corollary 4.4, and the algorithm AscendMinimal is nothing but the application of the algorithm GreedyUpMinimal to L. Hence, Proposition 3.4 implies that AscendMinimal outputs the minimal integral minimizer $\underline{q}^*$of L, which is the minimal integral equilibrium price vector $\underline{p}^*$ by Theorem 2.3 (i). Moreover, the number of updates of the price vector in AscendMinimal is equal to $\|\underline{q}^* - p^\circ\|_\infty = \|\underline{p}^* - p^\circ\|_\infty$. □

Note that any algorithm that increases the price vector by a 0-1 vector in each iteration requires updates of the price vector at least $\|\underline{p}^* - p^\circ\|_\infty$ times. Hence, the algorithm AscendMinimal is the fastest among all iterative auction algorithms of this type, and the trajectory of the price vector is a "shortest" path from the initial vector p° to the minimal equilibrium $\underline{p}^*$.

In addition, since $\|\underline{p}^* - p^\circ\|_\infty \leq \max_{i \in N}\{a(i) - p^\circ(i)\}$ by Proposition 4.5, we can guarantee that the number of updates of p is at most $\max_{i \in N}\{a(i) - p^\circ(i)\}$; note that this bound can be computed in advance before executing the algorithm.

To find the maximal equilibrium price vector $\overline{p}^*$ instead of the minimal one, we consider another variant of the ascending auction algorithm called AscendMaximal, where Step 0 and Step 1 in AscendMinimal are replaced with the following:

Step 0: Set $p := p^\circ$, where $p^\circ \in \mathbb{Z}^n$ satisfies $p^\circ \leq \overline{p}^*$ (e.g., $p^\circ = \mathbf{0}$).
Step 1: Find the maximal minimizer $X \subseteq N$ of $L(p + \chi_X)$.

That is, a maximal X is found in Step 1 instead of a minimal X. This modification makes it possible to output the maximal equilibrium price vector $\overline{p}^*$.

Theorem 4.8. *If the initial vector p° in the algorithm* AscendMaximal *satisfies $p^\circ \leq \overline{p}^*$, the algorithm outputs $\overline{p}^*$ and the number of updates of the price vector is exactly equal to $\|\overline{p}^* - p^\circ\|_\infty$.*

Similarly to AscendMinimal and AscendMaximal, we can consider two variants of the descending auction algorithm called DescendMaximal and DescendMinimal, where the price vector is decreased by a 0-1 vector. Note that the algorithm DescendMaximal is the same as the descending auction algorithm in Ausubel [4].

Algorithm DescendMaximal
Step 0: Set $p := p^\circ$, where $p^\circ \in \mathbb{Z}^n$ satisfies $p^\circ \geq \overline{p}^*$ (e.g., $p^\circ = a$ with $a \in \mathbb{Z}^n$ given by (4.1)).
Step 1: Find the minimal minimizer $X \subseteq N$ of $L(p - \chi_X)$.
Step 2: If $X = \emptyset$, then output p and stop.
Step 3: Set $p := p - \chi_X$ and go to Step 1.

Theorem 4.9 ([4])**.** *Starting from an integral vector p° with $p^\circ \geq \overline{p}^*$, the algorithm* DescendMaximal *outputs $\overline{p}^*$ in a finite number of iterations.*

The algorithm DescendMinimal is obtained from DescendMaximal by replacing Step 0 and Step 1 with the following:

Step 0: Set $p := p^\circ$, where $p^\circ \in \mathbb{Z}^n$ satisfies $p^\circ \geq \underline{p}^*$ (e.g., $p^\circ = a$ with $a \in \mathbb{Z}^n$ given by (4.1)).
Step 1: Find the maximal minimizer $X \subseteq N$ of $L(p - \chi_X)$.

The algorithms DescendMaximal and DescendMinimal are nothing but the application of the algorithms GreedyDownMaximal and GreedyDownMinimal to the Lyapunov function. Hence, the next theorems follow from Propositions 3.6 and 3.7.

Theorem 4.10. *If the initial vector p° in the algorithm* DescendMaximal *satisfies $p^\circ \geq \overline{p}^*$, the algorithm outputs $\overline{p}^*$ and the number of updates of the price vector is exactly equal to $\|\overline{p}^* - p^\circ\|_\infty$.*

Theorem 4.11. *If the initial vector p° in the algorithm* DescendMinimal *satisfies $p^\circ \geq \underline{p}^*$, the algorithm outputs $\underline{p}^*$ and the number of updates of the price vector is exactly equal to $\|\underline{p}^* - p^\circ\|_\infty$.*

4.3. Two-Phase Auction Algorithms

An advantage of ascending and descending auction algorithms is that the price vector is updated monotonically, which is an important property from the viewpoint of auctions. They, however, have a drawback that the initial price vector should be a lower or upper bound for the integral equilibrium price vector $\underline{p}^*$ (or $\overline{p}^*$). In contrast, the following algorithms, which we call the *two-phase auction* algorithms, can start with any initial price vector and find an equilibrium. Therefore, the number of iterations can be small if we can choose an initial vector that is close to an equilibrium.

As we see below, a two-phase auction algorithm is an application of an ascending auction algorithm with an arbitrary initial vector, followed by a descending auction algorithm. We first present a variant of the two-phase auction algorithm obtained from the combination of AscendMinimal and DescendMinimal, which always outputs the minimal equilibrium price vector $\underline{p}^*$

Algorithm TwoPhaseMinMin

Step 0: Set $p := p^\circ$, where $p^\circ \in \mathbb{Z}^n$ is any vector (to be chosen appropriately in practice). Go to Ascending Phase.

Ascending Phase:

Step A1: Find the minimal minimizer $X \subseteq N$ of $L(p + \chi_X)$.

Step A2: If $X = \emptyset$, then go to Descending Phase.

Step A3: Set $p := p + \chi_X$ and go to Step A1.

Descending Phase:

Step D1: Find the maximal minimizer $X \subseteq N$ of $L(p - \chi_X)$.

Step D2: If $X = \emptyset$, then output p and stop.

Step D3: Set $p := p - \chi_X$ and go to Step D1.

To analyze the number of iterations required by TwoPhaseMinMin, we define

$$\eta(p, q) = \|p - q\|_\infty^+ + \|p - q\|_\infty^- \qquad (p, q \in \mathbb{Z}^n), \tag{4.2}$$

where

$$\|p-q\|_\infty^+ = \max_{i\in N} \max(0, p(i)-q(i)), \qquad \|p-q\|_\infty^- = \max_{i\in N} \max(0, -p(i)+q(i)).$$

The proof of the next theorem is given in Section 4.4.2.

Theorem 4.12. *Starting from any integral vector p°, the algorithm* TwoPhaseMinMin *terminates by outputting the minimal integral equilibrium price vector $\underline{p}^*$, and the number of updates of the price vector in the ascending phase is at most $\eta(p^\circ, \underline{p}^*)$ and that in the descending phase is at most $2\eta(p^\circ, \underline{p}^*)$.*

We can consider another variant of the two-phase auction algorithm, to be called TwoPhaseMinMax, which is the combination of AscendMinimal and DescendMaximal. That is, TwoPhaseMinMax is the algorithm obtained by replacing Step D1 in TwoPhaseMinMin with the following:

Step D1: Find the minimal minimizer $X \subseteq N$ of $L(p - \chi_X)$.

A version of the algorithm TwoPhaseMinMax specialized to valuation functions on $\{0,1\}^n$ coincides with the one in Sun and Yang [30]. TwoPhaseMinMax is also similar to the "global Walrasian tâtonnement algorithm" in Ausubel [4], which repeats ascending and descending phases until some equilibrium p^* is found, where p^* is not necessarily equal to $\underline{p}^*$ or $\overline{p}^*$. Our analysis shows that the global Walrasian tâtonnement algorithm terminates after only one ascending phase and only one descending phase; see also [30]. In other words, the behavior of the global Walrasian tâtonnement algorithm coincides with that of TwoPhaseMinMax.

Theorem 4.13. *Starting from any integral vector p°, the algorithm* TwoPhaseMinMax *terminates by outputting some integral equilibrium price vector p^*, and the number of updates of the price vector in the ascending phase is at most $\eta(p^\circ,\ p^*)$ and that in the descending phase is at most $2\eta(p^\circ,\ p^*)$.*

The proof of this theorem is given in Section 4.4.3.

Remark 4.14. We may also consider other two-phase auction algorithms TwoPhaseMaxMin and TwoPhaseMaxMax. The former consists of AscendMaximal and DescendMinimal, and the latter consists of AscendMaximal and DescendMaximal. It can be shown that TwoPhaseMaxMin (resp., TwoPhaseMaxMax) finds the minimal integral equilibrium price vector $\underline{p}^*$ (resp., the maximal integral equilibrium price vector $\overline{p}^*$); the proof is similar to that for Theorem 4.12 and omitted. □

Remark 4.15. We point out that the iterative auction algorithms considered in this section use *linear* and *anonymous* pricing rule, meaning that the price of any bundle x of goods is equal to $p^\top x$ and is the same for all bidders. On the other hand, so-called combinatorial auction algorithms use *nonlinear* and *discriminatory* pricing rule, i.e., the price $p(x, i)$ of a bundle x of goods depends on x and bidder i and is nonlinear. It is shown that various iterative auction algorithms using the latter pricing rule can be used to find (possibly nonlinear and discriminatory) equilibrium prices even if valuation functions are more general than those with SGS condition (see, e.g., [9, Chapter 2]).

Such iterative auction algorithms, however, are difficult to use in practice since we need to deal with exponential number of prices. $\square$

4.4. Proofs

4.4.1. Proof of Proposition 4.5.

Since all bidders' valuation functions f_j ($j = 1, 2, \ldots, m$) satisfy the condition (A1), they are M$^\natural$-concave functions by Theorem 4.1. For the proof of Proposition 4.5, we use the following property of M$^\natural$-concave functions.

Lemma 4.16. *Let $f : [\mathbf{0}, u]_{\mathbb{Z}} \to \mathbb{R}$ be an M$^\natural$-concave function. For $x, y \in [\mathbf{0}, u]_{\mathbb{Z}}$ with $x \leq y$ and $i \in N$ with $y(i) < u(i)$, it holds that*

$$f(x + \chi_i) - f(x) \geq f(y + \chi_i) - f(y).$$

Proof. For $y + \chi_i$, x, and $i \in \mathrm{supp}^+((y + \chi_i) - x)$, (M$^\natural$-EXC) implies that

$$f(y + \chi_i) + f(x) \leq f(y) + f(x + \chi_i)$$

since $\mathrm{supp}^-((y + \chi_i) - x) = \emptyset$. Hence, the claim of the lemma follows. $\square$

Let $p^* \in \mathbb{R}^n$ be an equilibrium price vector. Let $x_1^*, x_2^*, \ldots, x_m^*$ be an allocation such that $x_1^* + x_2^* + \cdots + x_m^* = u$ and $x_j^* \in D_j(p^*)$ for all $j \in M$.

We first show that $p^*(i) \geq 0$ for every $i \in N$. Since $x_1^*(i) + x_2^*(i) + \cdots + x_m^*(i) = u(i)$ and $x_{j'}^*(i) \leq u(i)$ for all $j' \in M$, there exists some $j \in M$ such that $x_j^*(i) < u(i)$, which, together with $x_j^* \in D_j(p^*)$, implies that

$$f_j(x_j^*) - (p^*)^\top x_j^* \geq f_j(x_j^* + \chi_i) - (p^*)^\top (x_j^* + \chi_i),$$

where $x_j^* + \chi_i \in [\mathbf{0}, u]_{\mathbb{Z}}$. This inequality can be rewritten as

$$p^*(i) \geq f_j(x_j^* + \chi_i) - f_j(x_j^*) \geq 0,$$

where the last inequality is by the monotonicity assumption (A0) for f_j.

We then show that $p^*(i) \leq a(i)$ for every $i \in N$. Since $x_1^*(i) + x_2^*(i) + \cdots + x_m^*(i) = u(i) > 0$ and $x_{j'}^*(i) \geq 0$ for all $j' \in M$, there exists some $j \in M$ such that $x_j^*(i) > 0$, which, together with $x_j^* \in D_j(p^*)$, implies that

$$f_j(x_j^*) - (p^*)^\top x_j^* \geq f_j(x_j^* - \chi_i) - (p^*)^\top (x_j^* - \chi_i),$$

where $x_j^* - \chi_i \in [\mathbf{0}, u]_{\mathbb{Z}}$. This inequality can be rewritten as

$$p^*(i) \leq f_j(x_j^*) - f_j(x_j^* - \chi_i) \leq f_j(\chi_i) - f_j(\mathbf{0}) \leq a(i),$$

where the second inequality is by Lemma 4.16 and the third by the definition (4.1) of $a(i)$.

4.4.2. Proof of Theorem 4.12 for Algorithm TwoPhaseMinMin.

The key of the proof is the following property of L$^\natural$-convex functions. Recall that the Lyapunov function is regarded as a function defined on $\mathbb{Z}^n$.

Lemma 4.17 ([21, Theorem 7.7])**.** *Let $g : \mathbb{Z}^n \to \mathbb{R}$ be an L$^\natural$-convex function. For every integral $p, q \in \mathbb{Z}^n$ with $\mathrm{supp}^+(p - q) \neq \emptyset$, it holds that*

$$g(p) + g(q) \geq g(p - \chi_X) + g(q + \chi_X),$$

where $X = \arg\max_{i \in N}\{p(i) - q(i)\}$.

We show several lemmas below, from which Theorem 4.12 follows. Let $\hat{p}$ be the price vector at the end of the ascending phase and $\check{p}$ be the output of the algorithm. Recall that $\underline{p}^*$ denotes the (unique) minimal integral equilibrium price vector, which is also the (unique) minimal minimizer of the Lyapunov function L by Theorem 2.3.

Lemma 4.18. *The vector $\hat{p}$ is the minimal vector in the set* $\arg\min\{L(p) \mid p \in \mathbb{Z}^n,\ p \geq p^\circ\}$ *and satisfies $\hat{p} \geq \underline{p}^*$. In addition, the number of updates of the price vector in the ascending phase is exactly equal to $\|\hat{p} - p^\circ\|_\infty$.*

Proof. The behavior of the ascending phase is the same as that of the algorithm AscendMinimal applied to the function $\hat{L} : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ defined as

$$\hat{L}(p) = \begin{cases} L(p) & (\text{if } p \geq p^\circ), \\ +\infty & (\text{otherwise}), \end{cases}$$

which is also an L$^\natural$-convex function. Theorem 4.7 implies that in the ascending phase the number of updates of the price vector p is equal to $\|\hat{p} - p^\circ\|_\infty$, and $\hat{p}$ is the minimal minimizer of the function $\hat{L}$, i.e., $\hat{p}$ is the minimal vector in $\arg\min\{L(p) \mid p \in \mathbb{Z}^n,\ p \geq p^\circ\}$.

We now prove $\hat{p} \geq \underline{p}^*$. Assume, to the contrary, that $\hat{p} \not\geq \underline{p}^*$. Then, we have $\mathrm{supp}^+(\underline{p}^* - \hat{p}) \neq \emptyset$, and therefore Lemma 4.17 implies that

$$L(\underline{p}^*) + L(\hat{p}) \geq L(\underline{p}^* - \chi_X) + L(\hat{p} + \chi_X), \tag{4.3}$$

where $X = \arg\max_{i \in N}\{\underline{p}^*(i) - \hat{p}(i)\}$. Since $\hat{p} + \chi_X \geq \hat{p} \geq p^\circ$ and $\hat{p} \in \arg\min\{L(p) \mid p \in \mathbb{Z}^n,\ p \geq p^\circ\}$, we have $L(\hat{p} + \chi_X) \geq L(\hat{p})$, which, together with (4.3), implies $L(\underline{p}^* - \chi_X) \leq L(\underline{p}^*)$, i.e., $\underline{p}^* - \chi_X \in \arg\min_{\mathbb{Z}} L$, a contradiction to the fact that $\underline{p}^*$ is a minimal minimizer of L. $\square$

Lemma 4.19. $\|\hat{p} - p^\circ\|_\infty \leq \eta(p^\circ,\ \underline{p}^*)$.

Proof. If $\underline{p}^* \geq p^\circ$, then $\underline{p}^* = \hat{p}$ by Lemma 4.18. Since $\|\underline{p}^* - p^\circ\|_\infty = \|\underline{p}^* - p^\circ\|_\infty^+$ and $\|\underline{p}^* - p^\circ\|_\infty^- = 0$, it holds that

$$\|\hat{p} - p^\circ\|_\infty = \|\underline{p}^* - p^\circ\|_\infty = \|\underline{p}^* - p^\circ\|_\infty^+ + \|\underline{p}^* - p^\circ\|_\infty^- = \eta(p^\circ,\ \underline{p}^*).$$

We then assume that $\mathrm{supp}^+(p^\circ - \underline{p}^*) \neq \emptyset$. This implies $\mathrm{supp}^+(\hat{p} - \underline{p}^*) \neq \emptyset$. Let $X = \arg\max_{i \in N}\{\hat{p}(i) - \underline{p}^*(i)\}$.

Claim: $\min_{i \in X}\{\hat{p}(i) - p^\circ(i)\} = 0$.
[Proof of Claim] By Lemma 4.17, it holds that

$$L(\hat{p}) + L(\underline{p}^*) \geq L(\hat{p} - \chi_X) + L(\underline{p}^* + \chi_X). \tag{4.4}$$

Since $\underline{p}^*$ is a minimizer of L, we have $L(\underline{p}^* + \chi_X) \geq L(\underline{p}^*)$, which, combined with (4.4), implies $L(\hat{p} - \chi_X) \leq L(\hat{p})$. From this inequality follows that $\hat{p} - \chi_X \not\geq p^\circ$ since $\hat{p}$ is the minimal vector in the set $\arg\min\{L(p) \mid p \in \mathbb{Z}^n,\ p \geq p^\circ\}$ by Lemma 4.18. This concludes the proof since $\hat{p} - \chi_X \not\geq p^\circ$ holds if and only if $\min_{i \in X}\{\hat{p}(i) - p^\circ(i)\} = 0$. [End of Proof of Claim]

Let $t \in X$ be an element with $\hat{p}(t) = p^\circ(t)$. Then, it holds that

$$\begin{aligned} p^\circ(t) - \underline{p}^*(t) &= \hat{p}(t) - \underline{p}^*(t) \\ &= \max_{i \in N}\{\hat{p}(i) - \underline{p}^*(i)\} \\ &\geq \max_{i \in N}\{p^\circ(i) - \underline{p}^*(i)\} \\ &\geq p^\circ(t) - \underline{p}^*(t). \end{aligned}$$

Hence, all the inequalities in this formula hold with equality. In particular, we have

$$\max_{i \in N}\{\hat{p}(i) - \underline{p}^*(i)\} = \max_{i \in N}\{p^\circ(i) - \underline{p}^*(i)\} = \|\underline{p}^* - p^\circ\|_\infty^-,$$

where the last equality is by $\mathrm{supp}^+(p^\circ - \underline{p}^*) \neq \emptyset$. From this equation follows that for every $k \in N$, we have

$$\begin{aligned} \hat{p}(k) - p^\circ(k) &= [\underline{p}^*(k) - p^\circ(k)] + [\hat{p}(k) - \underline{p}^*(k)] \\ &\leq \|\underline{p}^* - p^\circ\|_\infty^+ + \max_{i \in N}\{\hat{p}(i) - \underline{p}^*(i)\} \\ &= \|\underline{p}^* - p^\circ\|_\infty^+ + \|\underline{p}^* - p^\circ\|_\infty^- = \eta(p^\circ,\ \underline{p}^*). \end{aligned}$$

Hence, $\|\hat{p} - p^\circ\|_\infty \leq \eta(p^\circ,\ \underline{p}^*)$ holds. $\square$

Lemma 4.20. *We have $\check{p} = \underline{p}^*$. In addition, the number of updates of the price vector in the descending phase is at most $2\eta(p^\circ,\ \underline{p}^*)$.*

Proof. Since $\hat{p} \geq \underline{p}^*$ holds by Lemma 4.18, the behavior of the descending phase is the same as that of the algorithm DescendMinimal applied to function L with initial vector $\hat{p}$. Hence, Theorem 4.11 implies that $\check{p} = \underline{p}^*$ and the number of updates of the price vector p in the descending phase is equal to $\|\underline{p}^* - \hat{p}\|_\infty$. We have

$$\begin{aligned}\|\underline{p}^* - \hat{p}\|_\infty &\leq \|\underline{p}^* - p^\circ\|_\infty + \|p^\circ - \hat{p}\|_\infty \\ &\leq \eta(p^\circ,\ \underline{p}^*) + \eta(p^\circ,\ \underline{p}^*) \\ &= 2\eta(p^\circ,\ \underline{p}^*),\end{aligned}$$

where the last inequality is by (4.2) and Lemma 4.19. □

Theorem 4.12 follows from Lemmas 4.18, 4.19, and 4.20 shown above.

4.4.3. Proof of Theorem 4.13 for Algorithm TwoPhaseMinMax*.*

Theorem 4.13 follows from Lemma 4.18 and Lemmas 4.21 and 4.22 below. Recall that p^* here denotes the output of the algorithm TwoPhaseMinMax.

Lemma 4.21. *The vector p^* is a minimizer of L and an integral equilibrium price vector. In addition, the number of updates of the price vector in the descending phase is at most $\eta(p^\circ,\ p^*) + \|p^\circ - \hat{p}\|_\infty$.*

Proof. The behavior of the descending phase is the same as that of the algorithm DescendMaximal applied to the function $\check{L} : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ defined as

$$\check{L}(p) = \begin{cases} L(p) & (\text{if } p \leq \hat{p}), \\ +\infty & (\text{otherwise}), \end{cases}$$

which is also an L$^\natural$-convex function. Theorem 4.10 implies that in the descending phase the number of updates of the price vector p is equal to $\|p^* - \hat{p}\|_\infty$, and p^* is the maximal minimizer of the function $\check{L}$, i.e., p^* is the maximal vector in $\arg\min\{L(p) \mid p \in \mathbb{Z}^n,\ p \leq \hat{p}\}$. Since $\underline{p}^* \leq \hat{p}$ holds by Lemma 4.18, we have

$$L(p^*) = \min\{L(p) \mid p \in \mathbb{Z}^n,\ p \leq \hat{p}\} = L(\underline{p}^*),$$

i.e., p^* is a minimizer of L. By Theorem 2.3 (i), p^* is an equilibrium price vector.

We have

$$\|p^* - \hat{p}\|_\infty \le \|p^* - p^\circ\|_\infty + \|p^\circ - \hat{p}\|_\infty \le \eta(p^\circ,\ p^*) + \|p^\circ - \hat{p}\|_\infty,$$

where the second inequality is by (4.2). □

Lemma 4.22. $\|\hat{p} - p^\circ\|_\infty \le \eta(p^\circ,\ p^*)$.

Proof. If $p^* = \hat{p}$, we have $\|p^* - p^\circ\|_\infty = \|p^* - p^\circ\|_\infty^+$ and $\|p^* - p^\circ\|_\infty^- = 0$, and therefore,

$$\|\hat{p} - p^\circ\|_\infty = \|p^* - p^\circ\|_\infty = \|p^* - p^\circ\|_\infty^+ + \|p^* - p^\circ\|_\infty^- = \eta(p^\circ,\ p^*).$$

If $p^* \ne \hat{p}$, we have $\mathrm{supp}^+(\hat{p} - p^*) \ne \emptyset$ since $p^* \le \hat{p}$. The rest of the proof is the same as that for Lemma 4.19, where $\underline{p}^*$ should be replaced with p^*. □

5. Connection to Unit-Demand Auction

The unit-demand auction model, where each bidder is interested in getting at most one item, is discussed extensively in the literature (see, e.g., [2, 3, 10, 18, 19, 28]). The model is known to be a special case of the general model with gross substitutes valuations considered in the previous sections. The objective of this section is to offer a unified treatment by showing that the general algorithms AscendMinimal, DescendMinimal, and TwoPhaseMinMin, when applied to the unit-demand auction model, coincide with existing fundamental iterative auction algorithms for the unit-demand auction model.

5.1. Unit-Demand Auction Model and Relationship with General Model

We explain the unit-demand auction model considered in this section, and show the relationship with the general auction model discussed in the previous sections.

The unit-demand auction model is a special case of the single-unit auction model, where each bidder is interested in getting at most one item, i.e., each bidder is a *unit-demand* bidder. This means that even if a bidder can get multiple items, the bidder is interested in only one item.

As in the previous sections, we denote by $N = \{1, 2, \ldots, n\}$ the set of items and by $M = \{1, 2, \ldots, m\}$ the set of bidders. We assume, without loss of generality, that each type of item is available in only one unit. For each item i and each bidder j, we denote by $v_j(i) \in \mathbb{Z}_+$ the valuation of item i

by bidder j. We define a valuation function $f_j : \{0,1\}^n \to \mathbb{Z}_+$ of bidder j by

$$f_j(x) = \begin{cases} \max\{v_j(i) \mid i \in \mathrm{supp}^+(x)\} & (\text{if } \mathrm{supp}^+(x) \neq \emptyset), \\ 0 & (\text{if } \mathrm{supp}^+(x) = \emptyset) \end{cases} \qquad (x \in \{0,1\}^n). \tag{5.1}$$

Through the one-to-one correspondence between 0-1 vectors and subsets of N, we identify the valuation function f_j in (5.1) defined on 0-1 vectors with the following set function defined on subsets of N:

$$f_j(X) = \begin{cases} \max\{v_j(i) \mid i \in X\} & (\text{if } X \neq \emptyset), \\ 0 & (\text{if } X = \emptyset). \end{cases} \tag{5.2}$$

A valuation function of this form is often called a *unit-demand valuation* (see, e.g., [9, Section 9.2.2] and [8, Definition 11.17]). It is easy to see that the unit-demand valuation function f_j in (5.1) satisfies the assumptions (A0) (i.e., f_j is monotone nondecreasing) and (A2) (i.e., f_j takes integer values). In addition, it is known that a unit-demand valuation is a typical example of gross substitutes valuation, i.e., f_j satisfies the assumption (A1).

Theorem 5.1 ([13]). *Valuation function $f_j : \{0,1\}^n \to \mathbb{R}$ given by* (5.1) *satisfies the GS condition (and also the SGS condition).*

Hence, the unit-demand auction model is a special case of the general model with gross substitutes valuations discussed in the previous sections, and all of the results there can be applied to the unit-demand auction model. To be consistent with (5.2), we rewrite the definition of the demand set $D_j(p)$ in (2.2) associated with a valuation function f_j in terms of set functions, i.e.,

$$D_j(p) = \{X \subseteq N \mid f_j(X) - p(X) \geq f_j(Y) - p(Y) \ (\forall Y \subseteq N)\},$$

where $p(Y) = \sum_{i \in Y} p(i)$ for $Y \subseteq N$.

Let $N^0 = N \cup \{0\}$, where 0 denotes an artificial item which has no value (i.e., $v_j(0) = 0$ for $j \in M$) and is available in infinite number of units. For each bidder j and a price vector $p \in \mathbb{R}^n_+$, we define a set $\widetilde{D}_j(p)$ by

$$\begin{aligned} \widetilde{D}_j(p) &= \arg\max\{v_j(i) - p(i) \mid i \in N^0\} \\ &= \{i \in N^0 \mid v_j(i) - p(i) \geq v_j(i') - p(i') \ (\forall i' \in N^0)\}, \end{aligned}$$

where we put $p(0) = 0$ for convenience. An *assignment* is a function $\pi : M \to N^0$, and an assignment π is said to be *feasible* if each item in N appears at most once in $\{\pi(j) \mid j \in M\}$ (the artificial item 0 may appear more than once).

A price vector $p^* \in \mathbb{R}^n$ is said to be a *Walrasian equilibrium price vector* if there exists a feasible assignment $\pi : M \to N^0$ such that $\pi(j) \in \widetilde{D}_j(p^*)$ for every $j \in M$ and $p^*(i) = 0$ for every item $i \in N \setminus \{\pi(j) \mid j \in M\}$. This definition of Walrasian equilibrium price vector is consistent with the definition given in Introduction in the case of unit-demand auction model. This fact, which seems to be well known among experts, is stated in the following proposition, where a proof is given in Appendix for completeness.

Proposition 5.2. *For a price vector $p \in \mathbb{R}^n_+$, there exists a feasible assignment $\pi : M \to N^0$ such that $\pi(j) \in \widetilde{D}_j(p)$ for every $j \in M$ and $p(i) = 0$ for every $i \in N \setminus \{\pi(j) \mid j \in M\}$ if and only if there exists a partition $\{X_1, \ldots, X_m\}$ of N such that $X_j \in D_j(p)$ (possibly $X_j = \emptyset$) for every $j \in M$.*

5.2. Review of Unit-Demand Auction Algorithms

In this section we review three iterative auction algorithms for the unit-demand auction model: Vickrey–English auction by Demange et al. [10], Vickrey–Dutch auction by Mishra and Parkes [18], and Vickrey–English–Dutch auction by Andersson and Erlanson [3]. The description of the algorithms given below basically follows [2] and [3]. In the following, we assume that valuation $v_j(i)$ for bidder j and item i is given by a nonnegative integer; this implies, in particular, that the valuation function f_j for $j \in M$ is an integer-valued function, and therefore there exists an integral equilibrium price vector by Theorem 2.3 and Corollary 4.4.

For a price vector $p \in \mathbb{R}^n_+$ and an item set $Y \subseteq N$, we define

$$O(Y, p) = \{j \in M \mid \widetilde{D}_j(p) \subseteq Y\},$$
$$U(Y, p) = \{j \in M \mid \widetilde{D}_j(p) \cap Y \neq \emptyset\}.$$

The set $O(Y, p)$ consists of bidders who only demand items in Y at price p, while $U(Y, p)$ is the set of bidders who demand some item in Y at price p. Obviously, $O(Y, p) \subseteq U(Y, p)$.

A set $Y \subseteq N$ is said to be *overdemanded* if $|O(Y, p)| > |Y|$. This condition means that there exists at least one bidder in $O(Y, p)$ who can get no item in $\widetilde{D}_j(p) \setminus \{0\}$. A set $X \subseteq N$ is said to be *in excess demand* at price p if it satisfies

$$|U(Y, p) \cap O(X, p)| > |Y| \qquad (\emptyset \neq \forall Y \subseteq X).$$

This means that for every nonempty subset Y of X, there exists at least one bidder in $U(Y, p) \cap O(X, p)$ who cannot get an item in Y. The following property of sets in excess demand is shown in [19, Proposition 1] (see also [3, Proposition 1] and [2, Theorem 1]).

Proposition 5.3. *Sets in excess demand at price p are closed under union operation, i.e., if $X, Y \subseteq N$ are in excess demand at price p, then $X \cup Y$ is also in excess demand at price p. In particular, a maximal set in excess demand at price p is uniquely determined.*

The Vickrey-English auction algorithm due to Mo et al. [19] and Sankaran [28], which is a variant of the one in Demange et al. [10], is described as follows:

Algorithm Vickrey_English
Step 0: Set $p := p^\circ$, where $p^\circ \in \mathbb{Z}^n_+$ satisfies $p^\circ \leq \underline{p}^*$ for the minimal equilibrium price vector $\underline{p}^*$ (e.g., $p^\circ = \mathbf{0}$).
Step 1: Find the maximal set $X \subseteq N$ in excess demand at price p.
Step 2: If $X = \emptyset$, then output p and stop.
Step 3: Set $p := p + \chi_X$ and go to Step 1. □

To describe the Vickrey-Dutch auction algorithm, we need variants of the sets $\widetilde{D}_j(p)$ and $O(Y, p)$ by taking the positivity of prices (i.e., positive or zero) into account as follows:

$$\widetilde{D}_j^+(p) = \widetilde{D}_j(p) \cap \mathrm{supp}^+(p) \quad (j \in M),$$
$$O^+(Y, p) = \{j \in M \mid \widetilde{D}_j^+(p) \subseteq Y\}.$$

A set $X \subseteq N$ is said to be *in positive excess demand* at price p if it satisfies $X \subseteq \mathrm{supp}^+(p)$ and

$$|U(Y, p) \cap O^+(X, p)| > |Y| \qquad (\emptyset \neq \forall Y \subseteq X).$$

The following proposition can be proved in a similar way.

Proposition 5.4 (cf. [3, Theorem 2])**.** *Sets in positive excess demand at price p are closed under union operation. In particular, a maximal set in positive excess demand at price p is uniquely determined.*

Vickrey–Dutch auction by Mishra and Parkes [18] is described as follows:

Algorithm Vickrey_Dutch
Step 0: Set $p := p^\circ$, where $p^\circ \in \mathbb{Z}^n_+$ satisfies $p^\circ \geq \underline{p}^*$ for the minimal equilibrium price vector $\underline{p}^*$.
Step 1: Find the maximal set $Z \subseteq N$ in positive excess demand at price p, and put $X = \mathrm{supp}^+(p) \setminus Z$.
Step 2: If $X = \emptyset$, then output p and stop.
Step 3: Set $p := p - \chi_X$ and go to Step 1. □

Vickrey–English–Dutch auction by Andersson and Erlanson [3], which is a combination of Vickrey–English auction and Vickrey–Dutch auction, is described as follows.

Algorithm VICKREY_ENGLISH_DUTCH
Step 0: Set $p := p^{\circ}$, where $p^{\circ} \in \mathbb{Z}^n$ is any vector (to be chosen appropriately in practice). Go to Step E1.
Step E1: Find the maximal set $X \subseteq N$ in excess demand at price p.
Step E2: If $X = \emptyset$, then go to Step D1.
Step E3: Set $p := p + \chi_X$ and go to Step E1.
Step D1: Find the maximal set $Z \subseteq N$ in positive excess demand at price p, and put $X = \mathrm{supp}^+(p) \setminus Z$.
Step D2: If $X = \emptyset$, then output p and stop.
Step D3: Set $p := p - \chi_X$ and go to Step D1.

5.3. Analysis of Unit-Demand Auction Algorithms

We first show that the unit-demand auction algorithms explained above coincide with the iterative auction algorithms in Section 4 applied to valuation functions f_j given by (5.1).

Theorem 5.5. *Let $f_j : \{0,1\}^n \to \mathbb{Z}_+$ be valuation functions given by* (5.1).
(i) *The sequence of price vectors p generated by the algorithm* VICKREY_ENGLISH *is the same as that of the algorithm* ASCENDMINIMAL *applied to valuation functions f_j.*
(ii) *The sequence of price vectors p generated by the algorithm* VICKREY_DUTCH *is the same as that of the algorithm* DESCENDMINIMAL *applied to valuation functions f_j.*
(iii) *The sequence of price vectors p generated by the algorithm* VICKREY_ENGLISH_DUTCH *is the same as that of the algorithm* TWOPHASEMINMIN *applied to valuation functions f_j.*

The crucial technical facts for the proof of Theorem 5.5 are the following connections between sets in excess demand and the Lyapunov function $L : \mathbb{Z}^n_+ \to \mathbb{R}$ in (2.5) associated with valuation functions f_j in (5.1).

Lemma 5.6. *Let $p \in \mathbb{Z}^n_+$ be a price vector.*
(i) *A set $X \subseteq N$ is the maximal set in excess demand at price p if and only if X is the minimal minimizer of $L(p + \chi_X) - L(p)$.*
(ii) *A set $Z \subseteq \mathrm{supp}^+(p)$ is the maximal set in positive excess demand at price p if and only if $X = \mathrm{supp}^+(p) \setminus Z$ is the maximal minimizer of $L(p - \chi_X) - L(p)$.*

The proof of Lemma 5.6 is given in Section 5.4.

A combination of Theorem 5.5 above and Theorems 4.7, 4.11, and 4.12 in Section 4 yields the following (exact or upper) bounds on the number of iterations in the unit-demand auction algorithms. The claims (i) and (ii) given below are already shown in [3, Corollary 2], while (iii) is a new result. Recall the definition of $\eta(p, q)$ in (4.2).

Corollary 5.7.
(i) *The number of updates of the price vector in the algorithm* VICKREY_ENGLISH *is exactly equal to* $\|\underline{p}^* - p^\circ\|_\infty$.
(ii) *The number of updates of the price vector in the algorithm* VICKREY_DUTCH *is exactly equal to* $\|\underline{p}^* - p^\circ\|_\infty$.
(iii) *The number of updates of the price vector in the algorithm* VICKREY_ENGLISH_DUTCH *is at most* $3\eta(p^\circ, \underline{p}^*)$.

5.4. Proof of Lemma 5.6

5.4.1. Proof of Lemma 5.6 (i).

We first prove Lemma 5.6 (i).

Lemma 5.8. *Let* $j \in M$. *For* $p \in \mathbb{Z}^n_+$ *and* $X \subseteq N$, *it holds that*

$$L(p + \chi_X) - L(p) = |X| - |O(X, p)|. \tag{5.3}$$

Proof. We first show the following equation:

$$V_j(p) = \max_{i \in N^0} \{v_j(i) - p(i)\}. \tag{5.4}$$

We have

$$\begin{aligned} V_j(p) &= \max\{f_j(X) - p(X) \mid X \subseteq N\} \\ &\geq \max\,[f_j(\emptyset) - p(\emptyset), \max\{f_j(\{i\}) - p(\{i\}) \mid i \in N\}] \\ &= \max\,[v_j(0) - p(0), \max\{v_j(i) - p(i) \mid i \in N\}] = \max_{i \in N^0}\{v_j(i) - p(i)\}. \end{aligned}$$

On the other hand, for every nonempty $X \subseteq N$, we have $v_j(i^*) - p(i^*) \geq f_j(X) - p(X)$, where $i^* \in X$ is an item with $v_j(i^*) = f_j(X)$. We also have $v_j(0) - p(0) = f_j(\emptyset) - p(\emptyset)$, from which follows that

$$V_j(p) = \max\{f_j(X) - p(X) \mid X \subseteq N\} \leq \max_{i \in N^0}\{v_j(i) - p(i)\}.$$

Hence, (5.4) holds.

From (5.4) follows that

$$V_j(p+\chi_X) - V_j(p) = \begin{cases} -1 & (\text{if } j \in O(X,p)), \\ 0 & (\text{otherwise}). \end{cases}$$

Therefore, it holds that

$$L(p+\chi_X) - L(p) = |X| + \sum_{j=1}^{m} \{V_j(p+\chi_X) - V_j(p)\} = |X| - |O(X,p)|.$$

$\square$

To establish a connection between the minimal minimizer of $L(p+\chi_X) - L(p)$ and the maximal set in excess demand, it is convenient to use a directed graph G defined as follows. For a price vector $p \in \mathbb{Z}^n_+$, we consider a directed graph $G = (V, E)$ with the vertex set $V = \{s,t\} \cup M \cup N^0$ and the edge set

$$E = \{(s,j) \mid j \in M\} \cup \{(i,t) \mid i \in N^0\} \cup \{(j,i) \mid j \in M,\ i \in \widetilde{D}_j(p)\}.$$

For each edge $(u,v) \in E$, we define its capacity $c(u,v)$ as

$$c(u,v) = \begin{cases} 1 & (\text{if } u = s \text{ and } v \in M, \text{ or } u \in N \text{ and } v = t), \\ +\infty & (\text{otherwise}). \end{cases}$$

Note that edge set E is dependent on price vector p. A vertex partition $(S, V \setminus S)$ with $s \in S$ and $t \in V \setminus S$ is called an *s-t cut*, and its the *capacity* $c(S, V \setminus S)$ is defined as

$$c(S, V \setminus S) = \sum_{(u,v) \in E(S, V \setminus S)} c(u,v),$$

where $E(S, V \setminus S) = \{(u,v) \in E \mid u \in S,\ v \in V \setminus S\}$. An s-t cut of G is said to be *minimum* if it has the minimum capacity among all s-t cuts.

For $X \subseteq N$, define a vertex set $K(X)$ by

$$K(X) = \{s\} \cup O(X,p) \cup X.$$

Note that $(K(X), V \setminus K(X))$ is an s-t cut. The next lemma shows that the capacity of $(K(X), V \setminus K(X))$ minus m $(= |M|)$ is equal to the right-hand side of (5.3), and that a minimum s-t cut is given by $(K(X), V \setminus K(X))$ for some X.

Lemma 5.9. *Let $X \subseteq N$.*
(i) *For every s-t cut $(S, V \setminus S)$ with $S \cap M \subseteq O(X,p)$ and $S \cap N^0 = X$, we have*

$$c(S, V \setminus S) = |X| - |S \cap M| + m.$$

In particular, we have

$$c(K(X), V \setminus K(X)) = |X| - |O(X,p)| + m.$$

(ii) *For every s-t cut $(S, V \setminus S)$ with $S \neq K(X)$ and $S \cap N^0 = X$, it holds that $c(K(X), V \setminus K(X)) < c(S, V \setminus S)$.*

Proof. We first prove (i). By the definition of edge set E, there exists no edge from $O(X,p) = \{j \in M \mid \widetilde{D}_j(p) \setminus X = \emptyset\}$ to $N^0 \setminus X$. Hence, it holds that

$$\begin{aligned}
c(S, V \setminus S) &= \sum_{(u,t) \in E,\ u \in X} c(u,t) + \sum_{(s,v) \in E,\ v \in M \setminus S} c(s,v) \\
&= |X| + |M \setminus S| \\
&= |X| - |S \cap M| + m.
\end{aligned}$$

We then prove (ii). Suppose that there exists some $j \in S \cap M$ such that $\widetilde{D}_j(p) \setminus X \neq \emptyset$. For $i \in \widetilde{D}_j(p) \setminus X$, we have $(j,i) \in E$ and $i \in V \setminus S$, and therefore

$$c(S, V \setminus S) \geq c(j,i) = +\infty.$$

If there exists no $j \in S \cap M$ with $\widetilde{D}_j(p) \setminus X \neq \emptyset$, we have $S \cap M \subseteq O(X,p)$. Moreover, $S \cap M \neq O(X,p)$, since $S \neq K(X)$. Using claim (i) above we obtain that

$$\begin{aligned}
c(S, V \setminus S) &= |X| - |S \cap M| + m \\
&> |X| - |O(X,p)| + m \\
&= c(K(X), V \setminus K(X)).
\end{aligned}$$

□

It follows from Lemmas 5.8 and 5.9 that a set $X \subseteq N$ is a minimizer of the value $L(p + \chi_X) - L(p)$ if and only if $(K(X), V \setminus K(X))$ is a minimum s-t cut of the graph G.

Our next step is to relate minimal such X to the maximal set in excess demand.

Lemma 5.10. *Let $X \subseteq N$ be the (uniquely determined) minimal set such that $(K(X), V \setminus K(X))$ is a minimum s-t cut of the graph G. Then, X is the maximal set in excess demand at price p.*

Proof. We first show that X is in excess demand at price p, i.e., $|U(Y,p) \cap O(X,p)| > |Y|$ holds for every nonempty $Y \subseteq X$. Putting $X' = X \setminus Y$ and $Z = U(Y,p) \cap O(X,p)$, we have

$$|O(X',p)| = |O(X,p)| - |Z|.$$

Hence, it follows that

$$\begin{aligned} c(K(X'), V \setminus K(X')) &= |X'| - |O(X',p)| + |M| \\ &= (|X| - |Y|) - (|O(X,p) - |Z|) + |M| \\ &= c(K(X), V \setminus K(X)) + (|Z| - |Y|), \end{aligned}$$

where Lemma 5.9 (i) is used. Since $(K(X), V \setminus K(X))$ is an s-t cut with the minimum capacity and $X' \subsetneq X$, the minimality of X implies $|Z| > |Y|$. Hence, X is in excess demand at price p.

To show that X is the unique maximal set among all sets in excess demand, we assume, to the contrary, that there exists some set $\tilde{X} \supsetneq X$ in excess demand (cf. Proposition 5.3). Since $\tilde{X}$ is in excess demand and $\tilde{X} \setminus X \neq \emptyset$, it holds that

$$|U(\tilde{X} \setminus X, p) \cap O(\tilde{X}, p)| > |\tilde{X} \setminus X|.$$

Putting $Z' = U(\tilde{X} \setminus X, p) \cap O(\tilde{X}, p)$, we also have $O(X,p) = O(\tilde{X}, p) \setminus Z'$. Hence, it follows that

$$\begin{aligned} c(K(\tilde{X}), V \setminus K(\tilde{X})) &= |\tilde{X}| - |O(\tilde{X},p)| + |M| \\ &= (|X| + |\tilde{X} \setminus X|) - (|O(X,p)| + |Z'|) + |M| \\ &< |X| - |O(X,p)| + |M| = c(K(X), V \setminus K(X)), \end{aligned}$$

a contradiction to the fact that $(K(X), V \setminus K(X))$ is a minimum s-t cut of G. Therefore, X is the unique maximal set among all sets in excess demand. □

From the discussion above, we see that a set $X \subseteq N$ is the minimal minimizer of the value $L(p + \chi_X) - L(p)$ if and only if X is the maximal set in excess demand at price p. Thus, Lemma 5.6 (i) holds.

5.4.2. Proof of Lemma 5.6 (ii).

The proof of Lemma 5.6 (ii) given below is similar to that for Lemma 5.6 (i).

Lemma 5.11. *Let $j \in M$. For $p \in \mathbb{Z}^n_+$, $Z \subseteq \mathrm{supp}^+(p)$, and $X = \mathrm{supp}^+(p) \setminus Z$, it holds that*

$$\begin{aligned} L(p - \chi_X) - L(p) &= -|X| - |O^+(\mathrm{supp}^+(p) \setminus X, p)| + |M| && (5.5)\\ &= |Z| - |O^+(Z, p)| - |\mathrm{supp}^+(p)| + |M|. && (5.6) \end{aligned}$$

Proof. We have $V_j(p) = \max_{i \in N^0}\{v_j(i) - p(i)\}$ by (5.4). Therefore, it holds that

$$V_j(p - \chi_X) - V_j(p) = \begin{cases} +1 & (\text{if } \widetilde{D}^+_j(p) \cap X \neq \emptyset), \\ 0 & (\text{otherwise}), \end{cases}$$

from which follows that

$$\begin{aligned} L(p - \chi_X) - L(p) &= -|X| + \sum_{j=1}^{m} \{V_j(p - \chi_X) - V_j(p)\} \\ &= -|X| + |\{j \in M \mid \widetilde{D}^+_j(p) \cap X \neq \emptyset\}| \\ &= -|X| + |\{j \in M \mid \widetilde{D}^+_j(p) \not\subseteq \mathrm{supp}^+(p) \setminus X\}| \\ &= -|X| + |M \setminus O^+(\mathrm{supp}^+(p) \setminus X, p)| \\ &= -|X| - |O^+(\mathrm{supp}^+(p) \setminus X, p)| + |M| \\ &= |Z| - |O^+(Z, p)| - |\mathrm{supp}^+(p)| + |M|. \end{aligned}$$

This concludes the proof. □

To relate the minimal minimizer X of $L(p - \chi_X) - L(p)$ with the maximal set in positive excess demand, we use a directed graph $G^+ = (V^+, E^+)$ with the vertex set $V^+ = \{s, t\} \cup M \cup \mathrm{supp}^+(p)$ and the edge set

$$E^+ = \{(s, j) \mid j \in M\} \cup \{(i, t) \mid i \in \mathrm{supp}^+(p)\} \cup \{(j, i) \mid j \in M, \ i \in \widetilde{D}^+_j(p)\}.$$

That is, the graph G^+ is a subgraph of the graph G defined in Section 5.4.1 obtained by removing vertices in $N^0 \setminus \mathrm{supp}^+(p)$. We define the capacity $c(u, v)$ for each edge $(u, v) \in E^+$ as in Section 5.4.1, i.e.,

$$c(u, v) = \begin{cases} 1 & (\text{if } u = s \text{ and } v \in M, \text{ or } u \in \mathrm{supp}^+(p) \text{ and } v = t), \\ +\infty & (\text{otherwise}). \end{cases}$$

For $Z \subseteq N$, a vertex set $K^+(Z)$ is defined by

$$K^+(Z) = \{s\} \cup O^+(Z, p) \cup Z.$$

Note that $(K^+(Z), V^+ \setminus K^+(Z))$ is an s-t cut. The next lemma shows that the capacity of $(K^+(Z), V^+ \setminus K^+(Z))$ minus $|\mathrm{supp}^+(p)|$ is equal to the right-hand sides of (5.5) and (5.6) with $X = \mathrm{supp}^+(p) \setminus Z$, and that a minimum s-t cut is given by $(K^+(Z), V^+ \setminus K^+(Z))$ for some Z.

Lemma 5.12. *Let $Z \subseteq \mathrm{supp}^+(p)$.*
(i) *For every s-t cut $(S, V^+ \setminus S)$ with $S \cap M \subseteq O^+(Z, p)$ and $S \cap \mathrm{supp}^+(p) = Z$, we have*

$$c(S, V^+ \setminus S) = |Z| - |S \cap M| + m.$$

In particular, we have

$$c(K^+(Z), V^+ \setminus K^+(Z)) = |Z| - |O^+(Z, p)| + m.$$

(ii) *For every s-t cut $(S, V^+ \setminus S)$ with $S \neq K^+(Z)$ and $S \cap \mathrm{supp}^+(p) = Z$, it holds that $c(K^+(Z), V^+ \setminus K^+(Z)) < c(S, V^+ \setminus S)$.*

Proof. The proof is essentially the same as that for Lemma 5.9 and therefore omitted. □

It follows from Lemmas 5.11 and 5.12 that a set $X \subseteq \mathrm{supp}^+(p)$ is a minimizer of the value $L(p - \chi_X) - L(p)$ if and only if $(K^+(Z), V^+ \setminus K^+(Z))$ is a minimum s-t cut of the graph G^+ for $Z = \mathrm{supp}^+(p) \setminus X$.

Our next step is to relate minimal such Z to the maximal set in positive excess demand.

Lemma 5.13. *Let $Z \subseteq N$ be the (uniquely determined) minimal set such that $(K(Z), V^+ \setminus K(Z))$ is a minimum s-t cut of the graph G^+. Then, Z is the maximal set in positive excess demand at price p.*

Proof. The proof is quite similar to that for Lemma 5.10 and therefore omitted. □

From the discussion above, the following equivalence holds for $X \subseteq \mathrm{supp}^+(p)$ and $Z = \mathrm{supp}^+(p) \setminus X$:

$$\begin{aligned} & X \text{ is the maximal minimizer of the value } L(p - \chi_X) - L(p) \\ \iff \; & Z \text{ is the minimal set such that } (K(Z), V^+ \setminus K(Z)) \text{ is} \\ & \qquad\qquad\qquad\qquad \text{a minimum } s\text{-}t \text{ cut of the graph } G^+ \\ \iff \; & Z \text{ is the maximal set in positive excess demand at price } p. \end{aligned}$$

Thus, Lemma 5.6 (ii) holds.

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References

[1] T. Andersson, C. Andersson, A.J.J. Talman, Sets in excess demand in simple ascending auctions with unit-demand bidders, Working Paper No. 2010:15 (revised June 2012), Department of Economics, Lund University, 2010.

[2] T. Andersson, C. Andersson, A.J.J. Talman, Sets in excess demand in simple ascending auctions with unit-demand bidders, Ann. Oper. Res. 211 (2013) 27–36.

[3] T. Andersson, A. Erlanson, Multi-item Vickrey–English–Dutch auctions, Games Econ. Behav. 81 (2013) 116–129.

[4] L. M. Ausubel, An efficient dynamic auction for heterogeneous commodities, Amer. Econ. Rev. 96 (2006) 602–629.

[5] L. M. Ausubel, P. Milgrom, Ascending auctions with package bidding, Front. Theor. Econ. 1 (2002) Article 1.

[6] M. Bichler, P. Shabalin, A. Pikovsky, A computational analysis of linear price iterative combinatorial auction formats, Inform. Syst. Res. 20 (2009) 33–59.

[7] M. Bing, D. Lehmann, P. Milgrom, Presentation and structure of substitutes valuations, in: Proc. 5th ACM Conf. Electronic Commerce (EC04), ACM, New York, 2004, pp. 238–239.

[8] B. Blumrosen, N. Nisan, Combinatorial auction, in: N. Nisan, T. Roughgarden, É. Tardos, V. V. Vazirani (Eds.), Algorithmic Game Theory, Cambridge Univ. Press, Campbridge, 2007, pp. 267–299.

[9] P. Cramton, Y. Shoham, R. Steinberg, Combinatorial Auctions, MIT Press, Cambridge, 2006.

[10] G. Demange, D. Gale, M. Sotomayor, Multi-item auctions, J. Polit. Econ., 94 (1986) 863–872.

[11] S. Fujishige, Submodular Functions and Optimization, Second Edition, Elsevier, Amsterdam, 2005.

[12] S. Fujishige, Z. Yang, A note on Kelso and Crawford's gross substitutes condition, Math. Oper. Res. 28 (2003) 463–469.

[13] F. Gul, E. Stacchetti, Walrasian equilibrium with gross substitutes, J. Econ. Theory 87 (1999) 95–124.

[14] F. Gul, E. Stacchetti, The English auction with differentiated commodities, J. Econ. Theory 92 (2000) 66–95.

[15] A. S. Kelso, Jr., V.P. Crawford, Job matching, coalition formation and gross substitutes, Econometrica 50 (1982) 1483–1504.

[16] B. Lehmann, D. Lehmann, N. Nisan, Combinatorial auctions with decreasing marginal utilities, Games Econ. Behav. 55 (2006) 270–296.

[17] P. Milgrom, B. Strulovici, Substitute goods, auctions, and equilibrium, J. Econ. Theory 144 (2009) 212–247.

[18] D. Mishra, D.C. Parkes, Multi-item Vickrey–Dutch auctions, Games Econ. Behav. 66 (2009) 326–347.

[19] J.-P. Mo, P.-S. Tsai, S.-C. Lin, Pure and minimal overdemanded sets: a note on Demange, Gale and Sotomayor, Unpublished Mimeo, 1988.

[20] K. Murota, Convexity and Steinitz's exchange property, Adv. Math. 124 (1996) 272–311.

[21] K. Murota, Discrete Convex Analysis, SIAM, Philadelphia, 2003.

[22] K. Murota, On steepest descent algorithms for discrete convex functions, SIAM J. Optim. 14 (2003) 699–707.

[23] K. Murota, A. Shioura, M-convex function on generalized polymatroid, Math. Oper. Res. 24 (1999) 95–105.

[24] K. Murota, A. Shioura, Exact bounds for steepest descent algorithms of L-convex function minimization, Oper. Res. Lett. 42 (2014) 361–366.

[25] K. Murota, A. Shioura, A. Yang, Computing a Walrasian equilibrium in iterative auctions with multiple differentiated items, in: Proc. 24th Int. Symp. Algorithms and Computation (ISAAC13), LNCS 8283, Springer, Berlin, 2013, pp. 468–478.

[26] K. Murota, A. Tamura, New characterizations of M-convex functions and their applications to economic equilibrium models with indivisibilities, Discrete Appl. Math. 131 (2003) 495–512.

[27] D.C. Parkes, L.H. Ungar, Iterative combinatorial auctions: theory and practice, in: Proc. 17th National Conference on Artificial Intelligence (AAAI00), AAAI Press/MIT Press, Cambridge, 2000, pp. 74–81.

[28] J.K. Sankaran, On a dynamic auction mechanism for a bilateral assignment problem, Math. Social Sci. 28 (1994) 143–150.

[29] A. Shioura, A. Tamura, Gross substitutes condition and discrete concavity for multi-unit valuations: a survey, J. Oper. Res. Soc. Japan 58 (2015) 61–103.

[30] N. Sun, Z. Yang, A double-track adjustment process for discrete markets with substitutes and complements, Econometrica 77 (2009) 933–952.

[31] W. Vickrey, Counterspeculation, auctions, and competitive sealed tenders, J. Finance 16 (1961) 8–37.

Appendix A. Proof of Proposition 5.2

We first prove the "if" part. Let p be a price vector, and suppose that there exists a partition $\{X_1, \dots, X_m\}$ of N such that $X_j \in D_j(p)$ (possibly $X_j = \emptyset$) for all $j \in M$.

Claim: For each $j \in M$, the following properties hold:
(i) The set X_j in the partition contains at most one item in $\mathrm{supp}^+(p)$.
(ii) If $X_j \cap \mathrm{supp}^+(p) \neq \emptyset$, then the unique item i^* in $X_j \cap \mathrm{supp}^+(p)$ satisfies $i^* \in X_j \cap \widetilde{D}_j(p)$.
(iii) If $X_j \neq \emptyset$, then $X_j \cap \widetilde{D}_j(p) \neq \emptyset$.
(iv) If $X_j = \emptyset$, then $0 \in \widetilde{D}_j(p)$.
[Proof of Claim (i)] For $i \in X_j \cap \mathrm{supp}^+(p)$, we have $v_j(i) = f_j(X_j)$ since otherwise (i.e., if $v_j(i) < f_j(X_j)$) we have

$$f_j(X_j) - p(X_j) = f_j(X_j \setminus \{i\}) - p(X_j) < f_j(X_j \setminus \{i\}) - p(X_j \setminus \{i\}), \quad \text{(A.1)}$$

a contradiction to the fact that $X_j \in D_j(p)$. If $|X_j \cap \mathrm{supp}^+(p)| \geq 2$, then we have $f_j(X_j \setminus \{i\}) = f_j(X_j)$, and therefore the inequality (A.1) holds again, a contradiction.

[Proof of Claim (ii)] For the unique item i^* in $X_j \cap \mathrm{supp}^+(p)$ we have $v_j(i^*) = f_j(X_j)$ and $p(X_j) = p(i^*)$. Since $X_j \in D_j(p)$, it holds that

$$\begin{aligned} v_j(i^*) - p(i^*) &= f_j(X_j) - p(X_j) \\ &\geq \max[f_j(\emptyset) - p(\emptyset), \max_{i \in N}\{f_j(\{i\}) - p(\{i\})\}] \\ &= \max[v_j(0) - p(0), \max_{i \in N}\{v_j(i) - p(i)\}]. \end{aligned} \tag{A.2}$$

Hence $i^* \in \widetilde{D}_j(p)$.

[Proof of Claim (iii)] By Claim (ii), we may assume $X_j \cap \mathrm{supp}^+(p) = \emptyset$ with $X_j \neq \emptyset$. Let i^* be an item in X_j with the maximum value of $v_j(i^*)$. Then, we have $v_j(i^*) = f_j(X_j)$ and $p(i^*) = p(X_j) = 0$. Hence, the inequality (A.2) holds again, i.e., $i^* \in \widetilde{D}_j(p)$. This shows $X_j \cap \widetilde{D}_j(p) \neq \emptyset$.

[Proof of Claim (iv)] If $X_j = \emptyset$, then $v_j(0) - p(0) = 0 = f_j(X_j) - p(X_j)$. Hence, (iv) follows. [End of Claim]

Based on the claims above, we define an assignment $\pi : M \to N^0$ as follows:

$$\pi(j) = \begin{cases} \text{the unique item in } X_j \cap \mathrm{supp}^+(p) & (\text{if } X_j \cap \mathrm{supp}^+(p) \neq \emptyset), \\ \text{any item in } X_j \cap \widetilde{D}_j(p) & (\text{if } X_j \cap \mathrm{supp}^+(p) = \emptyset \text{ and } X_j \neq \emptyset), \\ 0 & (\text{otherwise (i.e., if } X_j = \emptyset)). \end{cases}$$

We show that this assignment satisfies the conditions in the statement of Proposition 5.2.

Since $\{X_1, \ldots, X_m\}$ is a partition of N and $\pi(j) \in X_j$ holds whenever $\pi(j) \neq 0$, we have $\pi(j) \neq \pi(j')$ for every $j, j' \in M$ with $\pi(j) \neq 0$ and $\pi(j') \neq 0$. Hence, the assignment π is a feasible assignment. By Claims (ii), (iii), and (iv), $\pi(j) \in \widetilde{D}_j(p)$ holds for all $j \in M$. Since each X_j contains at most one item with positive price, we have $p(i) = 0$ for $i \in N \setminus \{\pi(j) \mid j \in M\}$. This concludes the proof of "if" part.

Next we prove the "only if" part of the statement. Assume that there exists a feasible assignment $\pi : M \to N^0$ such that $\pi(j) \in \widetilde{D}_j(p)$ for all $j \in M$ and $p(i) = 0$ for every item $i \in N \setminus \{\pi(j) \mid j \in M\}$. We may assume that $\pi(j) \neq 0$ for some $j \in M$; if $\pi(j) = 0$ for all $j \in M$, then we may set $\pi(1) = 1$ since $v_1(1) \geq 0 = v_1(0)$.

Let $\{X_1, \ldots, X_m\}$ be any partition of N satisfying the condition that

$$\text{if } \pi(j) \neq 0 \text{ then } \pi(j) \in X_j, \qquad \text{if } \pi(j) = 0 \text{ then } X_j = \emptyset.$$

Since $\pi(j) \neq 0$ for some $j \in M$, such a partition exists. For $j \in M$ and $i \in X_j \setminus \{\pi(j)\}$, we have $p(i) = 0$.

It remains to show that $X_j \in D_j(p)$ holds for each $j \in M$. We first consider the case with $\pi(j) \neq 0$. Since $\pi(j) \in \widetilde{D}_j(p) \setminus \{0\}$ and $p(i) = 0$ for $i \in X_j \setminus \{\pi(j)\}$, we have

$$v_j(\pi(j)) \geq v_j(\pi(j)) - p(\pi(j)) \geq v_j(i) - p(i) = v_j(i) \qquad (\forall i \in X_j \setminus \{\pi(j)\}).$$

Hence, we have $v_j(\pi(j)) = f_j(X_j)$ and $p(\pi(j)) = p(X_j)$, which implies that

$$f_j(X_j) - p(X_j) = v_j(\pi(j)) - p(\pi(j)).$$

For any nonempty $Y \subseteq N$, let i' be an item in Y with $v_j(i') = f_j(Y)$. Then, it holds that

$$f_j(Y) - p(Y) = v_j(i') - p(Y) \leq v_j(i') - p(i').$$

Therefore,

$$f_j(X_j) - p(X_j) = v_j(\pi(j)) - p(\pi(j)) \geq v_j(i') - p(i') = f_j(Y) - p(Y).$$

We also have

$$f_j(X_j) - p(X_j) = v_j(\pi(j)) - p(\pi(j)) \geq v_j(0) - p(0) = f_j(\emptyset) - p(\emptyset).$$

Thus, $X_j \in D_j(p)$ holds.

We then consider the case with $\pi(j) = 0$. We have $0 \in \widetilde{D}_j(p)$ and $f_j(\emptyset) - p(\emptyset) = 0 = v_j(0) - p(0)$. Using this fact, we can prove $f_j(\emptyset) - p(\emptyset) \geq f_j(Y) - p(Y)$ for all $Y \subseteq N$ in a similar way as in the previous case. That is, $X_j = \emptyset \in D_j(p)$ holds.